TRIGONOMETRY

FOR BEGINNERS

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BY

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PREFACE.

TRIGONOMETRY is of all the subjects which can be classed as Mathematics the most practical, and yet in most elementary text-books it is explained in a manner which is purely theoretical. Of late years the question has been raised whether it is not possible to teach symbolical mathematics in such a way that the student is from the beginning able to attach a practical meaning to every symbol which he uses.

Experience shews that a mathematical subject taught from the theoretical side is almost repulsive to all but the very few boys who have the very exceptional gift of a taste for symbols, while practical surveying and other allied subjects are attractive to the much larger class who are endowed with the practical sense. It seems a pity that the elements of trigonometry, which are quite simple when once they are grasped, should be rendered distasteful to those to whom they would be most useful, by the way in which they are generally presented. The method of prefacing the teaching of theoretical geometry by a course of geometrical drawing is no doubt a step in the right direction, provided too much time is not given to a subject which, from a mathematical point of view, is only preliminary.

It seemed to the writers of the following pages that it ought to be possible to apply a similar method to Trigonometry.

Accordingly, an attempt has been made to provide an

introduction to Trigonometry which, while it is thoroughly sound, is based on examples all of which can be realized in thought, while many of them can be actually worked out in practice.

Thus it will be seen that the writers have endeavoured to provide a text-book which, while it is introductory to Higher Mathematics, will also give to engineering students that knowledge of Trigonometrical principles which is absolutely necessary for them. The first 144 pages form a treatise on Trigonometry for beginners which may be considered as an introduction to any elementary mathematical text-book on the subject.

To this an Appendix is added, giving an explanation of Circular Measure, and of the extension of the definitions of the Trigonometrical Ratios to angles of any magnitude and the relations between the Ratios of angles whose sum or difference is any multiple of $\pi/2$. This renders the book a text-book suitable for students preparing for the Examination in Stage II. Mathematics of the Board of Education.

Explanations are given of the principles and construction of certain instruments, viz. the Level, the Vernier, Sextant and the Theodolite. It is hoped that students will in all cases be able to handle the instruments themselves when reading the description.

Considerable use is made of accurate drawing to scale, and explanations, based upon such drawings, are given of the theory of the Trigonometrical Ratios, of Tables of their values, of their rate of increase, and of the principles of interpolation. With regard to this principle, 'the Theory of Proportional Parts' (as it is called) has often been to students merely a *rule* by

which certain results can be obtained and nothing more; a good example of the mere mechanical manipulation of figures without any intelligent grasp of the meaning of the process. The use of graphs, introducing the idea of a continuous curve, provides an explanation which may possibly help to clear up this difficulty. In the Table of Logarithms on pages 88, 89 we have given to *five* figures the logs. of numbers up to 3 figures and have left it to the student to interpolate. In this way we attain sufficient accuracy for all practical calculations (an accuracy which is not attained by four figure tables) with great economy of space and at the same time provide very instructive exercises for the beginner.

The book has been written in the hope of providing an intelligible introduction to Trigonometry and not as a cram book for an examination, although the standard of the examination of Stage II. of the Board of Education has to some extent suggested the limits to be aimed at.

We have to thank the Controller of His Majesty's Stationery Office for permission to print examination papers in Trigonometry set by the Board of Education.

We have also to thank Messrs Griffin, London (Figs. 16, 65), Messrs Davis, Derby and London (Figs. 66, 67), Messrs Stanley, London (Fig. 72) for permission to use reproductions from their catalogues and instruments.

We shall be grateful for any suggestions for the improvement of the work, and for any note of errors or misprints which may be discovered by those using it.

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TRIGONOMETRY FOR BEGINNERS.

INTRODUCTORY.

§ 1. The Symmetry of a Circle.

1. Take a piece of thin tough paper (or tracing linen) and on it draw a circle with centre **O** and any convenient radius; draw a diameter **AB**.

Fold the paper carefully, figure outwards, about the diameter **AB**. Hold the folded paper up to the light.

It will be found that the two parts into which the circumference of the circle is divided by **AB** coincide with one another; that is, the two semicircles are congruent.

Shew, by folding the paper about other diameters, that **any pair of semicircles are congruent.**

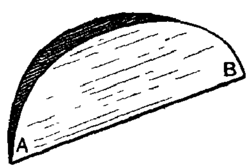


FIG. 1.

NOTE. *When a figure has been drawn on paper, and the paper* *to be folded about some line connected with the figure, it is usual* *say* "Fold the figure about the straight line." *A similar* *aning is attached to the instructions,* "Cut out the figure," "Place e triangle upon the other."

2. Draw two circles with equal radii, say 1·5 inches; cut the circles out and, by passing a pin through the centre of each, place them so that their centres coincide. What do you observe?

Can you say that **any two concentric circles of equal radii are congruent?**

3. Draw a circle with centre O and any convenient radius; draw a diameter AB. Mark with a pinhole the position of any point on the circumference.

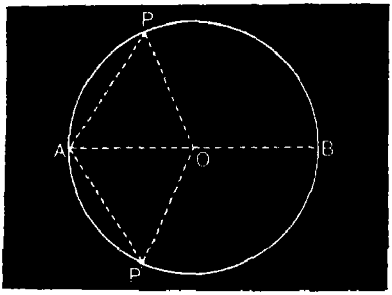


FIG. 2.

Fold the circle about AB, and with the pin mark the point (P′) upon which P falls. Open the paper out flat and join OP, OP′. [Fig. 2.]

Is P′ on the circumference of the circle? [Why?]

Is $\angle AOP = \angle AOP'$? [Why?]

Is chord AP = chord AP′? [Why?]

Is arc AP = arc AP′? [Why?]

Points, such as P and P′ in Experiment 3, and straight lines, which coincide when a figure is folded about a straight line, are called **corresponding points** and **corresponding straight lines.**

Thus, if ABCD is a square, B and D are corresponding points with regard to the diagonal AC, and AB, AD and CB, CD are corresponding lines with regard to this diagonal.

4. Fold the circle used in Expt. **3** about the line OP; mark the points (Q, R) upon which A, P′ fall respectively.

Are Q, R on the circumference? [Why?]

Is $\angle POQ = \angle QOR$? [Why?]

Is chord PQ=chord QR? [Why?]

Is arc PQ=arc QR? [Why?]

If a plane figure can be folded about a straight line so that the two parts, into which the figure is divided by the straight line, coincide, the figure is said to be **symmetrical with regard to the straight line**; the straight line is called an **axis of symmetry** of the figure, and any point or line in one of the parts is called the **image** or **reflection in the straight line** of the corresponding point or line in the other part.

Hence, it follows from Experiment **1**, that **a circle is symmetrical about any diameter.** In Fig. 2, **P′ is the image** (or reflection) **of P in the diameter AB.**

5. Draw a circle, and fold it about any chord which is not a diameter.

Is a circle symmetrical about any other line except a diameter?

6. Mark on a piece of paper a circle by running the point of a pencil round the edge of a penny.

Cut out the circle, and, by folding, crease the paper along axes of symmetry. It will be found that everyone of these axes passes through one point (the centre of the circle).

Hence, it follows that

(1) Every diameter of a circle is an axis of symmetry;

(2) Every axis of symmetry of a circle is a diameter.

7. Take any three points P, Q, R on a sheet of paper. Holding the paper up to the light, fold it (i) so that P and Q coincide, (ii) so that Q and R coincide, marking the crease in each case. Shew that the point in which the creases intersect is the centre of the circle passing through P, Q, R.

It follows from Expt. 7 that, if two circles have three points in common, they have the same point as centre, and therefore, by Expt. 2, are congruent.

Hence **two different circles cannot intersect in more than two points.**

Again two intersecting circles have different centres; and therefore the line passing through these centres is the only line which is a diameter of both circles; that is, **two intersecting circles have only one axis of symmetry, which is the line joining their centres.**

8. Draw a circle with centre O and any radius.

Take any point A on the circumference and with A as centre, and unequal radii, draw two arcs of circles cutting the first circle in P, P′ and Q Q′ respectively.

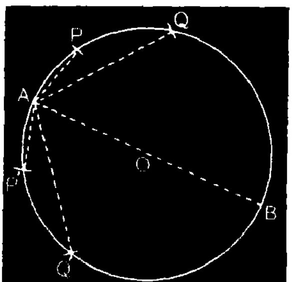


FIG. 3.

Then AP=AP′, and AQ=AQ′. [Why?]
In what line are P′, Q′ the images of P, Q? [Why?]
Shew that arc PQ=arc P′Q′ and ∠POQ=∠P′OQ′.

9. Verify, by folding, that if P, Q, P′, Q′ are four points on the circumference of a circle whose centre is O, and

(i) if $\angle POQ = \angle P'OQ'$; then chord PQ = chord P′Q′, and arc PQ = arc P′Q′;

(ii) if arc PQ = arc P′Q′; then chord PQ = chord P′Q′, and $\angle POQ = \angle P'OQ'$;

(iii) if chord PQ = chord P′Q′; then arc PQ = arc P′Q′, and $\angle POQ = \angle P'OQ'$.

The results obtained in Expt. 9 are very important. By means of them we are enabled to devise instruments and methods for measuring and reproducing angles.

§ 2. Protractors.

In order to measure numerical values of angles, a unit angle must first be decided upon.

In Geometry the unit used is the **right angle.** It is convenient for the following reasons:—

(1) Its shape is familiar,

(2) Its definition is simple,

(3) It can easily be constructed,

(4) It is constant, for it can be shewn that all right angles are equal.

In Practical Trigonometry the unit chosen is the ninetieth part of a right angle; this angle is called a **degree.**

If the arc of a semicircle is divided into 180 equal parts, and the points of division are joined to the centre, there will be 180 equal angles at O. Hence, since there are 180 degrees in two right angles, each of these equal angles is a **degree.**

The degree is divided into 60 **minutes**, and the minute into 60 **seconds**.

The signs °, ′, ″ are used as abbreviations for "degrees," "minutes," "seconds." Thus we write $6° 23' 46''$ for "six degrees + twenty-three minutes + forty-six seconds."

We have therefore

$$1 \text{ rt.} \angle = 90°,$$
$$1° = 60',$$
$$1' = 60''.$$

The arc of a circle subtending at its centre an angle of 1° is only slightly greater than $\frac{1}{60}$th of the radius; so that it is impossible without using an inconveniently large radius to make a diagram shewing arcs corresponding to much smaller angles than 1°.

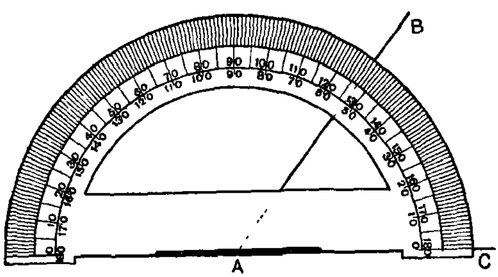


FIG. 4.

The usual form of the **circular protractor** (Fig. 4), in which the radius of the outer graduated semicircular arc is about 1·6 of an inch, reads to degrees, and by estimation to half a degree, the divisions being about ·03 of an inch apart.

A useful form is Low's 4-inch *quadrantal protractor* reading to half-degrees between 0° and 90°. It is made

of thin white celluloid; and angles may be estimated with it to a tenth of a degree with practice.

Circular protractors made of thin transparent celluloid, with the divisions marked on the underside, are also sold, and these are extremely handy and accurate.

10. Measure a given angle BAC [Fig. 4].

Place the protractor so that the vertex of the angle is just visible in the notch marking the centre of the semicircular arcs and one of the arms of the angle is in a direct line with the division marked 0°. then read off the number of degrees corresponding to the division which is in a direct line with the other arm of the angle.

Thus in Fig. 4, the angle BAC is 53° (approx.).

The circular protractor may also be used to draw an angle equal to a given angle.

11. Draw an angle of 53° with a circular protractor.

Method (*a*). In Fig. 4. Suppose AC to be one of the arms of the angle to be drawn: place the protractor so that the central notch is at A, and bring the division marked 0° into a direct line with AC; draw a *short straight line* in a direct line with the line joining A to the point on the arc corresponding to 53°: remove the protractor and draw the arm AB passing through A and the mark you have made.

Method (*b*). Suppose AB is the arm of the angle first drawn: place the protractor so that the central notch is at A and bring the point on the arc corresponding to 53° directly over the line AB: then, using the diametral edge of the protractor as a ruler, draw the other arm AC. [It must be noticed that for this method the little feet of the protractor in Fig. 4 should be absent, i.e., the straight edge of the protractor must coincide with the division marked 0°.]

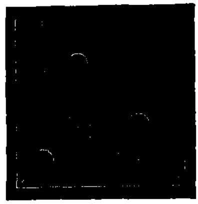


Fig 5

The method (*b*) is the principle of an improved protractor designed by Professor Low [Fig. 5]. It consists of two parts made of white celluloid; a segmental part **DE**, the circular edge of which has a tongue which fits into a corresponding groove on the right-angled part as shewn. The circular edge of **DE** is graduated in degrees, and a mark on the circular edge of **ABC**, shown in Fig. 5, in coincidence with 45° on **DE**; so that the straight or drawing edge of **DE** may be set to any angle from 0° to 90° with **AB** or **BC**. The effective diameter of the protractor is 7 inches

12. On a piece of tough stout paper or thin cardboard draw a rectangle, 6 in. long and $1\frac{1}{2}$ in broad, and mark the middle point of one of the longer sides. Place the circular protractor with its diametral edge along this side, the notch being at the middle point.

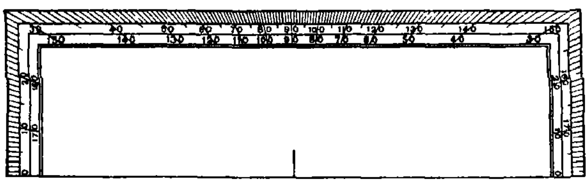


Fig. 6.

Mark off with a needle-point, or a well-sharpened hard pencil, the divisions on the semicircular arc from 0° to 180°. Remove the protractor, and join each of the marks to the mark made at the middle point of the side of the rectangle; draw four straight lines parallel to the other three sides of the rectangle and ink in, as shewn in Fig. 6.

A **rectangular protractor** has been constructed; it may be cut out and kept for use.

For constructing angles of given magnitude the rectangular protractor should be used as in method (*b*) for the circular protractor.

13. Construct, with the rectangular protractor, an angle of 37°.

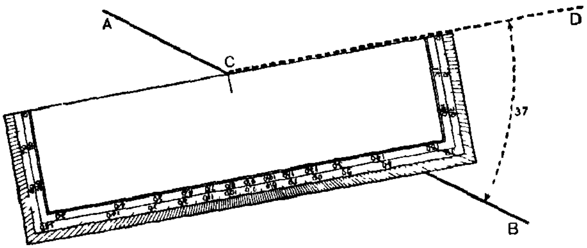


FIG 7.

Take a straight line AB, and C any point in it Place the protractor so that the notch is at C, and the division for 37° is in a direct line with CB. Then using the edge of the protractor as a ruler draw CD the other arm of the angle BCD. Then BCD is 37°.

Note 1. The usual method involves the following operations —(i) placing the notch at a point, (ii) placing the edge in coincidence with a straight line, (iii) marking a point opposite a division, and (iv) joining two points The operations (iii) and (iv) are the most liable to error and these are eliminated in the above method, which is therefore quicker and more accurate

Note 2 The rectangular protractor can be looked upon as a circular protractor of varying radius, the divisions being further apart for angles from 0° to 45°, but nearer together from 45° to 90° It should always be very carefully tested before being used for any drawing requiring any great accuracy

If a given angle, drawn on a sheet of paper, has to be copied upon another sheet of paper it can be "pricked off" either (*a*) with three points, one of which is at the vertex of the angle, or more accurately, (*b*) with four points, two on each arm of the angle.

The sheet on which the angle is drawn should be laid perfectly flat upon the other, and the "pricker" should be held vertically whilst it is pushed through both sheets. If there is not a pricker amongst the drawing instruments, one may easily be made by mounting a fine needle in a handle. Frequently, however, on unscrewing the handle of the drawing-pen in a box of instruments, a needle will be found mounted in the handle.

If the given angle is drawn on material that is not thin or cannot be pierced, or if the copy has to be drawn on the same surface or in some special position, the ordinary Geometrical construction, depending on the property found in Expt. 8 (iii), must be used.

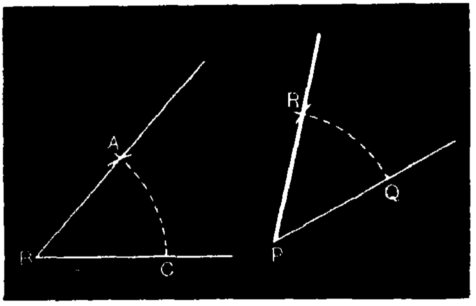


FIG. 8.

Thus, in Fig. 8, if **ABC** is the given angle, and an angle equal to it has to be drawn on the upper side of **PQ**, with its vertex at **P**, we proceed as follows:—With centre **B** and any radius describe a circle cutting the arms of the angle in **C**, **A**: with centre **P** and the same radius draw an arc **QR**: with centre **Q** and radius equal to *chord* **CA**, draw an arc intersecting the arc **QR** in **R**: join **PR**.

Then $\angle RPQ = \angle ABC$. [Why?]

If we decide always to use a constant radius for the first arc, the radius for the second arc (i.e. the *chord* subtending the given angle), will also be constant for any

given angle. Hence, these chords can be determined once for all from a circular protractor and a **scale of chords** made. This scale, if of sufficiently large "radius," can be used for measuring and constructing angles with great accuracy: being in the form of a scale it is also more handy than either form of protractor; it should always be used when great accuracy is necessary.

14. Draw OA, OB, two straight lines at right angles. With centre O and radius = 4″ describe an arc cutting the straight lines in A and B.

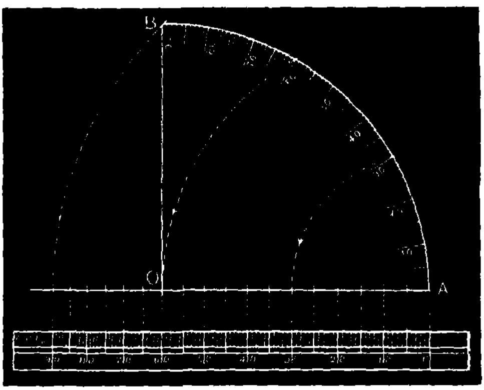


FIG. 9.

Graduate the arc in degrees with the circular protractor. With centre A and radii equal to the chords corresponding to 1°, 2° ... 90° respectively, describe arcs cutting AO. A **scale of chords** has been constructed, which may be inked in and figured, as in Fig. 9, and cut out for use.

Note. The lengths of the chords may be tabulated and a **table of chords** formed, which is fairly accurate and handy in use.

Exercises.

1. Given a scale of chords, how do you find the radius of the circle from which it has been constructed?

2. How many degrees are there in the angle between the hands of a watch at 4 o'clock?

3. How many degrees does (i) the hour hand, (ii) the minute hand rotate through in one minute?

4. What is the angle between the hands of a watch at (i) 4.25; (ii) 5.40; (iii) 9.7?

5. Draw six triangles of different shapes and sizes, and measure with the *circular protractor* the sum of the three angles in each triangle.

6. Draw four quadrilaterals of different shapes and sizes, and find by means of the *scale of chords* the sum of the four angles of each.

7. Express $\frac{5}{27}$ of a right angle in degrees, minutes and seconds.

8. How often does the angle $59^\circ 44' 42''$ contain the angle $2^\circ 50' 42''$?

9. One angle of a triangle is $50^\circ 0' 27''$; if one of the other angles is double of the third, find the other two angles.

Note. **The sum of the interior angles of any rectilinear figure of n sides is $2n-4$ *right angles*.**

10. Using either protractor or scale of chords, draw regular polygons of 4, 5, 6, 8, and 10 sides.

§ 3. Points of the Compass.

Sailors, in reckoning the direction of a ship's motion, use an eighth of a right angle as unit angle; this angle is called a **point**, For sub-divisions they use **half-points** and **quarter-points**.

A Mariner's Compass Card is shewn in Fig. 10.

Fig. 10.

The names of the points are derived from those of the four cardinal points as follows:

N.W. (Nor'-West) bisects the angle between North and West. S.S.E. (Sou'-Sou'-East) bisects the angle between South and South-East. "By" is short for "altered by"; thus S. by E. indicates a direction South altered by *one point* towards the East; W.N.W. by $\frac{3}{4}$ W., a direction bisecting the angle between West and North-West altered by *three-quarters of a point towards* the West, and is the same as W. by N. by $\frac{1}{4}$ N., making an angle of about 14° with West.

1. How many degrees are there in a point?

2. Find the angle between the directions, (i) W. by N. and N.N.W., (ii) S.E. by E. and E.N.E.

3. Draw a straight line ON, where N is north of O, and make a diagram shewing the positions of objects whose distances and directions from O are given as:—

B, 2 miles, E. by S. by $\frac{3}{4}$ S.; C, 3 miles, S.S.E; D, 4 miles, S.W. by $\frac{3}{4}$ S.; E, $2\frac{1}{2}$ miles, W. by N.; F, $3\frac{1}{2}$ miles, N.N.W. by $\frac{1}{4}$ W.

Measure the angles of the polygon BCDEF.

The direction in which an object lies, as viewed from a place in the same horizontal plane, is called its **bearing.** If the bearing of an object cannot be exactly expressed by points of the compass, degrees are used.

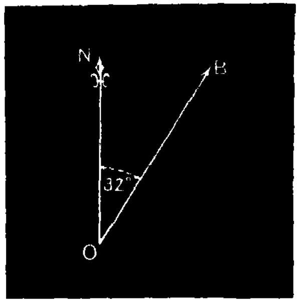


Fig. 11.

Thus, in Fig. 11, the bearing of B from O can be expressed either as

$32°$ E. of N., $58°$ N. of E.,

or N. by $32°$ E., E. by $58°$ N.

4. The bearing of O from B is S. by $32°$ W. or W. by $58°$ S [Draw a figure to prove this statement.]

5. Find the distance and bearing of each of the points C, D, E, F from B, in Ex. 3, p. 13.

6. As I steer a ship, I see a red, and a white, light, which I know (from a chart) proceed from two lighthouses, the one showing the red light being 10 miles due north of the other. The red light bears N. by $25°$ W., the white W. by $23°$ N. Draw a figure representing the position of the ship and measure its distance from the red light.

7. A ship steaming N.N.W. at 18 knots an hour, is due west of a lighthouse at 2 a.m.; the light bears S.E. at 2.37 a.m. Draw a figure, shewing the lighthouse and the track of the ship; and measure the distance and bearing of the lighthouse from the ship at 3 a.m.

8. As I walked along a straight road in a direction E.N.E., I noticed two churches A and B, bearing N.N.E., and N.E. by N., respectively. These churches came into line 300 yds. further along; at the end of another 200 yds. the bearing of B was W. by $25°$ N., whilst A was hidden by a hill. Find the distance of A from B.

In general practical work it is not necessary to find the bearing of a single object, but the difference in bearing of two objects, i.e. **the angle subtended by the objects** (or the straight line joining them) **at the observer's eye,** if the objects and the observer's eye are all at the same level.

If this is not the case, the difference of bearing is the angle subtended by the projections of the objects upon a **horizontal plane** through the eye.

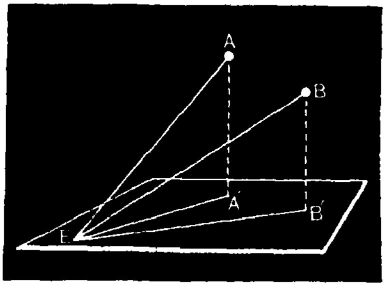


Fig. 12.

Thus, in Fig. 12, let A and B represent two objects; let AA′, BB′ be **vertical** (as ascertained by a plumb-line); let E be the observer's eye, and EA′, EB′ perpendicular to AA′, BB′: then EA′, EB′ are **horizontal lines**, and the plane containing EA′, EB′ is a **horizontal plane.** The difference of bearing of A, B as seen from E is $\angle$A′EB′, whilst the angle subtended by A, B at E is $\angle$AEB, which is *not equal* to $\angle$A′EB′ as a general rule.

The angle AEA′ is called the **elevation** or (angular) **altitude** of A as seen from E.

If a horizontal line AC is drawn, the angle **CAE** is called the **depression** of E as seen from A.

9. If P, Q are two objects, shew that **the elevation of P viewed from Q is equal to the depression of Q viewed from P.**

§ 4. The Level and the Theodolite.

A **level** is used to ascertain whether a line is horizontal.

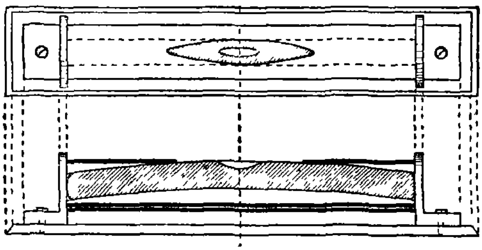


Fig. 13.

Fig. 13 shews the plan and sectional elevation of an ordinary* "carpenter's level." It consists of a closed circular cylindrical tube, nearly full of spirit, fitted into a wooden frame with a plane metal base. The tube, not being quite full, has a bubble of air left in it which occupies the highest point in the tube. The instrument is so constructed that when the bubble is bisected by a mark on the tube, seen through a little glass window in the top of the instrument, the base lies along a horizontal line in the direction of its length.

To ascertain whether a plane surface of a given object is horizontal, two observations of the position of the bubble when the base of the level is applied to the surface in different directions, are *necessary* and *sufficient*. [Why?]

* The figure shews an improved form in which the frame is all metal, and has a revolving tube, which closes and protects the glass tube when not in use.

Another form of level is commonly used on Physical instruments and photographic cameras. It consists of

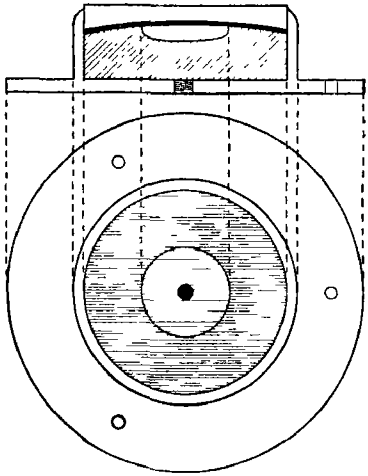


FIG. 14.

a shallow cylindrical metal box, into which is fitted a glass cover with its lower surface spherical. The box is *nearly* filled with spirit, leaving room for a bubble of air which, as in the ordinary level, occupies the highest point. It is so arranged that, when the base of the level is accurately horizontal, the centre of the bubble coincides with the centre of the glass top of the instrument.

This form is extremely handy; for, by one application of the instrument to a plane surface, we can ascertain whether that surface is horizontal. On physical instruments and cameras a level of this form is often permanently fixed; the position of the bubble then indicates how the instrument has to be tilted to make its base horizontal; for the displacement of the bubble from its

central position is upwards along the line of greatest inclination to the horizon of the surface to which the level is fixed.

15. Take a piece of wood, $6'' \times \frac{3}{4}'' \times \frac{3}{4}''$, accurately rectangular, and fit a piece of stout cardboard or thin wood, $8'' \times 6''$, into a slit in it, so that the cardboard is accurately perpendicular to one of the long faces ED of the wood. In the central line of the cardboard $3''$ from the top, mark a point A; with A as centre and $3''$ radius describe a circle and draw a diameter CAB through A parallel to the face of the wood ED. Graduate the arc in degrees, the lines AB, AC indicating $0°$, and attach to the cardboard at A, by means of a long drawing pin, a cardboard pointer with two small drawing pins inserted, points outwards, near the ends (G).

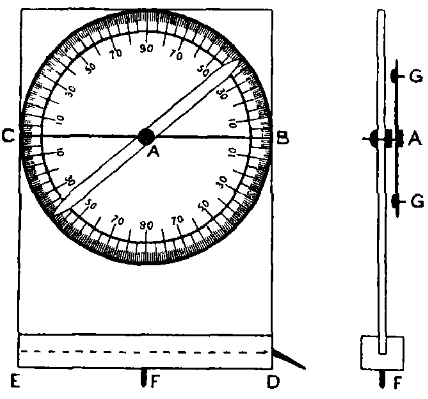


Fig. 15.

Use this instrument to obtain the elevation of objects in the schoolroom, such as the clock, the tops of the windows, placing it on a table or other plane surface, which has been "levelled."

What "horizontal" are you using?

16. Fasten a small spike in the wood (F in Fig. 15), directly under A, and fix a needle (D in Fig. 15) at one end of the wood so that it is in the plane of the cardboard but inclined to CD. On a piece of wood about $5''$ square, $\frac{1}{2}''$ thick, draw a

circle of 5″ radius and graduate the circumference in degrees; insert the spike F at the centre of this circle. You have now constructed an instrument by means of which you can simultaneously determine the difference of bearing and the angular elevations of two objects.

This is a simple model of an important instrument used in Surveying, called a **theodolite**. Where possible

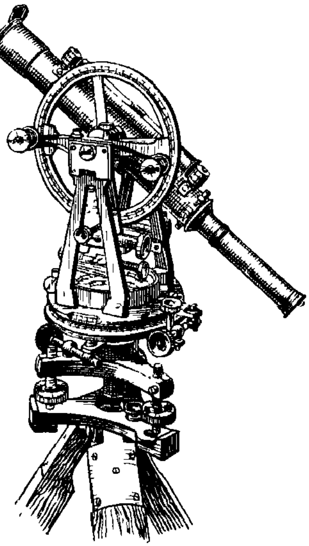


FIG. 16. The Theodolite*.

each student should make one of these models for himself in the school workshop, and "elementary practical surveying" should be practised with it in the playground, a circular level being fitted to the base board and a camera tripod used for a stand. A full description of the theodolite will be found on pages 127–130.

* From an instrument supplied by Messrs Griffin, London. This firm also supplies a reliable *model* theodolite for school use, which can be recommended.

§ 5. Graphical Solution of "Heights and Distances."

The sun is so far away from the earth that the "paths" of all those rays of sunlight which at any instant fall on different points within a small area are approximately parallel; that is, they all make the same angle with a horizontal plane.

This angle is called the **altitude of the sun** at that instant.

17. Set up a stump in the sunlight on a cricket pitch, being careful to get the stump perfectly vertical. [How?]

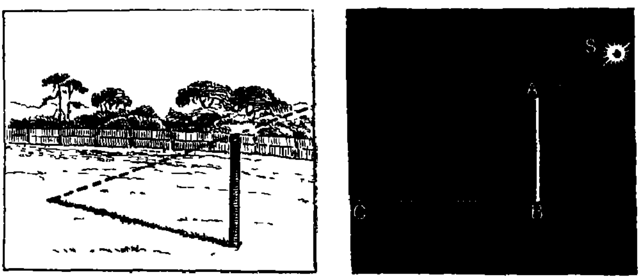


FIG. 17.

Measure the length of the part of the stump above the ground and of its shadow on the ground. Make a diagram ABC, in which B is a right angle and AB, BC *represent* the stump and its shadow, drawn to some convenient scale.

Measure the angle ABC; i.e. find the altitude of the sun at the instant.

1. If the Sun's Altitude is 42°, what is the length of the shadow cast by a walking-stick 3ft. 2ins. long?

2. If a stick a yard long casts a shadow 25 inches long, what is the Sun's Altitude? [R.F. $= \frac{1''}{10}$.]

3. The shadow of a poplar tree is 140 feet long; the shadow of a yard stick at the same time is 43 inches: find the height of the tree.

In question 3 above, a poplar tree was taken because such a tree grows nearly vertically. For, in finding the height of objects by a simultaneous determination of the length of its shadow and of the altitude of the sun, it is necessary to be able to measure the length of the shadow from its extremity to a point *vertically underneath the highest point of the object*, the measurement being made on a plane surface, or on *the same slope* as the measurements for determining the altitude of the sun.

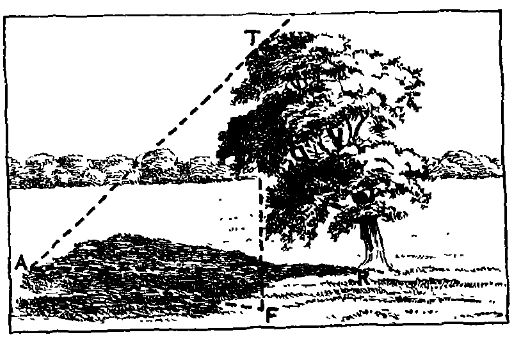


FIG. 18.

For instance, the length it would be necessary to measure in determining the height of the leaning tree in Fig. 18 is AF, where F is vertically under the highest point of the tree, (and not AR).

It would often be difficult to find accurately the *highest* point. In the figure as drawn T is not the highest point. It is the top of the tree as seen from A.

Again, in measuring the height of a hill it is, in general, impossible to get at the foot of the vertical line through the highest point.

Moreover the shadow will not usually fall conveniently for the purpose of measurement on a horizontal plane, so the elevation of the top of the object must be found at a convenient *station*, irrespective of the sun's altitude, by means of a theodolite.

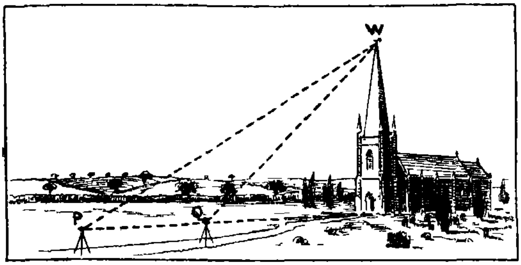
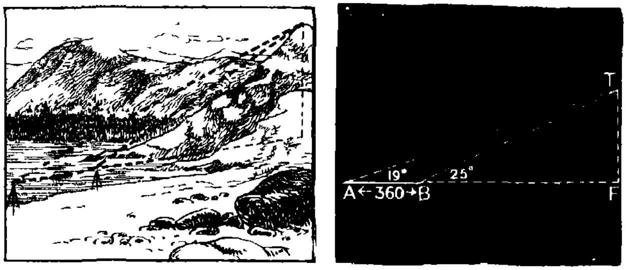


Fig. 19.

For example, to find the height of a weathercock W, we could by means of our model theodolite find the elevation of W at a station P, and, proceeding *straight towards* the vertical through W for a ft, find the elevation of W at Q. If P and Q are in a horizontal line or inclined to the horizontal plane at a known angle, then the angles P and Q and the distance PQ are sufficient data to draw the triangle PQW: from which the height can be found.

Even on very rough ground it may be possible to find isolated positions, here and there, which are in the same horizontal plane as the foot of the object whose height we wish to find. If two of these happen to be in a direct line with the object and their distance apart can be determined, then the height of the object can at once be found.

18. Fig. 20 (*a*) represents a hill whose height above a certain lake is required.



(*a*) (*b*)

Fig. 20.

Two convenient stations A and B are found on the margin of the lake. AB is measured and the elevations of T, as seen at A and B, are observed. It is found that AB = 120 yds., elevation at A = 19°, and elevation at B = 25°: hence in the triangle ABT the side AB, the interior angle at A and the exterior angle at B are known, and the triangle can be constructed as follows:—

Draw a st. line ABF. Mark off AB = 1·8 in. [R.F. = $\frac{1}{2400}$]; make $\angle$FAT = 19° and $\angle$FBT = 25°; draw TF $\perp$ AF; measure TF.

It will be found that TF = 2·38 in.

$\therefore$ Height of the hill = $2{\cdot}38 \times \frac{2400}{12}$ ft.

= 476 ft.

Note.—This height is the height of T above the horizontal line through the pivot (A in the figure of the model theodolite of Expt. 16) of the telescope of the theodolite. The height of this above the level of the lake must be added to obtain the height of the hill above the level of the lake.

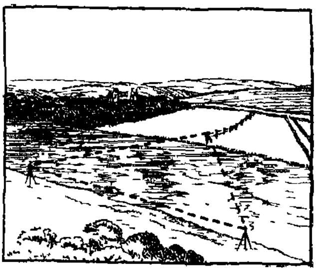
It is obvious that, if the distance between A and B is known (say they are consecutive milestones on a straight horizontal road), the angle of depression of each may be observed from T instead of the angles of elevation of T from A and B.

4. From the top of a cliff 130 feet high the angles of depression of two boats are 57° and 49°: if they are in a direct line with the foot of the cliff, find the distance between them. [Draw a sketch as well as the diagram to scale.]

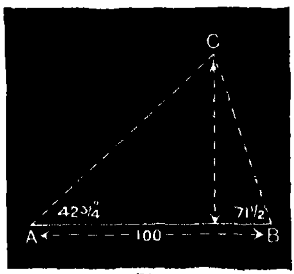
5. A flagstaff on top of a cliff, and at its edge, is 35 feet high. From a ship at sea the angles of elevation of the top and bottom of the pole are 16° 30′ and 18° respectively; find the distance of the ship from the cliff and the height of the cliff. [Draw a sketch as well as the diagram to scale.]

6. From the top of a lighthouse the depression of a ship at sea is 42°: from a window 23 feet lower down the depression is 36°. Find the distance of the vessel from the foot of the lighthouse, and the height of the lighthouse.

19. Fig. 21 (*a*) represents a river, whose width is required.



(*a*)



(*b*)

Fig. 21.

Two positions A and B on one bank are chosen and from the position A the difference in bearing of an object C, on the opposite bank, and of the other position B is observed: from B the difference in bearing of C and A is observed; AB is measured; i.e. AB and the angles at A and B of the triangle ABC are known and the perpendicular distance of C from AB can be found.

Find the breadth of the river given that AB = 100 yds., $\angle A = 42\frac{3}{4}°$, $\angle B = 71\frac{1}{2}°$. [R.F. = $\frac{1}{800}$.]

7. As I walk along a straight road, I notice a church whose direction makes an angle of 18° with the road; one mile further on its direction makes an angle of 72°; what is the shortest distance of the church from the road?

8. If the spire of the church in question 7 is 200 ft. high, find its elevation at the two positions at which the church was observed.

9. A sphere of lead, one inch in diameter, is attached to a thread thirty inches long to form a pendulum: the pendulum vibrates through 10° on each side of the vertical. Find the greatest height to which the centre of the lead sphere rises, and the greatest displacement horizontally, assuming that the centre of the sphere always lies in a straight line with the string.

10. A fire-escape cannot get nearer the wall of a burning house than 15 feet. How long must the escape be to reach a window-sill 65 feet above the ground? What angle will the escape make with the ground?

11. A person, 5 ft. 10 in. high, walks under and past a street lamp fixed to a wall. When he is 28 feet from the point vertically under the lamp, his shadow is 20 ft. long: find the height of the lamp.

12. A telegraph pole, 50 ft. high, is prevented from bending by a wire attached to ring-bolts at the top and bottom of the pole and a horizontal stretching-rod 4 ft. long fixed to the pole at a height of 35 ft. Find the length of wire between the rings.

13. A flagstaff is fixed upright by four ropes attached to the top of the staff and to pegs in the ground 10 feet from the foot of the staff. If the flagstaff is 28 feet high, how much rope is required, allowing a foot at each end for tying?

14. A bridge is thrown across a ravine through which a stream 17 feet wide runs: the sides of the ravine slope upwards, right from the banks of the stream, at angles of 80° and 74° respectively. The bridge is 85 feet long. What is its height above the stream?

§ 6. Definitions of the Trigonometrical Ratios, Sine, Cosine and Tangent, for an acute angle.

20. Draw any acute angle BAC.

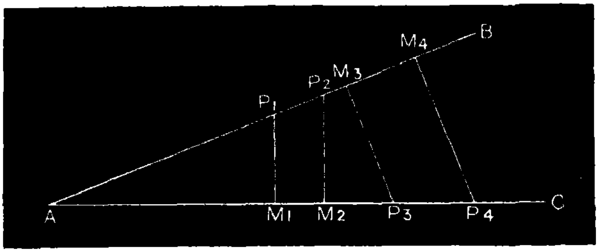


FIG. 22.

Take any number of points $M_1 M_2 M_3 \ldots$, in either arm, and draw $M_1 P_1$, $M_2 P_2$, $M_3 P_3 \ldots$ perpendicular to AM_1, AM_2, $AM_3 \ldots$, as in Fig. 22. Measure the lengths MP, AM, AP, and insert them in a table, as below.

	MP	AM	AP	$\frac{MP}{AP}$	$\frac{AM}{AP}$	$\frac{MP}{AM}$
1						
2						
3						
4						

Work out the ratios $\frac{MP}{AP}$, $\frac{AM}{AP}$, $\frac{MP}{AM}$ as *decimals*, and insert them in the last three columns respectively.

If the measurements are carefully made, the numbers in each of the last three columns will be very nearly equal.

NOTE. Do not take the point P too near A, or else the percentage error in measuring the lengths of the lines will be large.

The same result will be obtained for any acute angle; this fact can be expressed thus:

If BAC is any given acute angle, M *any* point in either arm and MP at right angles to that arm, the values of each of the fractions

$$\frac{MP}{AP}, \quad \frac{AM}{AP}, \quad \frac{MP}{AM}$$

are constant for all positions of P.

These fractions, or ratios, are called respectively the **sine, cosine** and **tangent** of the angle **A**: the abbreviations **sin A, cos A, tan A** being used.

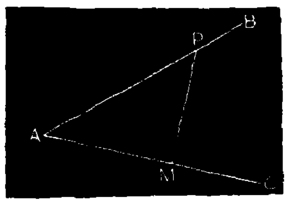


Fig. 23.

Thus $\sin A = \dfrac{MP}{AP}$.

$\cos A = \dfrac{AM}{AP}$,

$\tan A = \dfrac{MP}{AM}$; $= \dfrac{MP}{AM}$

or in terms of the angles of the *right-angled* triangle formed from a given angle,

$$\sin(\text{angle}) = \frac{\text{side opposite angle}}{\text{side opposite the right angle}},$$

$$\cos(\text{angle}) = \frac{\text{side opposite other acute angle}}{\text{side opposite the right angle}},$$

$$\tan(\text{angle}) = \frac{\text{side opposite angle}}{\text{side opposite other acute angle}}.$$

If in Fig. 23 the angle **APM** is considered as the given angle **P**, and **M** as any point in one arm and **MA** as drawn at right angles to that arm, then

$$\sin \mathsf{P} = \frac{\text{side opposite } \mathsf{P}}{\text{side opposite } \mathsf{M}} = \cos \mathsf{A},$$

$$\cos \mathsf{P} = \frac{\text{side opposite } \mathsf{A}}{\text{side opposite } \mathsf{M}} = \sin \mathsf{A},$$

$$\tan \mathsf{P} = \frac{\text{side opposite } \mathsf{P}}{\text{side opposite } \mathsf{A}} = \frac{1}{\tan \mathsf{A}}.$$

The angles **A** and **P**, whose sum is a right angle, are called **complementary angles.**

Hence

$$\textbf{sin (any angle)} = \textbf{cos (complementary angle)},$$

$$\frac{1}{\textbf{tan (any angle)}} = \textbf{tan (complementary angle)}.$$

21. Draw a quadrant of a circle with a radius 10 in. Draw two radii **OX**, **OY** at right angles, and graduate the arc **XY** in degrees. Through each point of division (**P**) draw a perpendicular (**PM**) to **OX**. Measure these perpendiculars true to ·01 in., divide each by 10 and thus construct a **table of sines** true to **three places of decimals.**

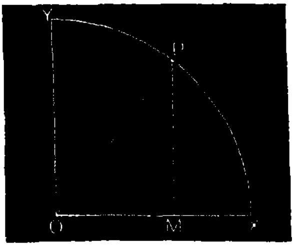


Fig. 24.

Compare the results with the tables on pp. 36, 37.

Construct from the same diagram a table of cosines and verify $\sin \mathsf{XOP} = \cos(90° - \mathsf{XOP})$.

22. Draw a quadrant of a circle with a radius 10 in. Draw two radii OX, OY at right angles, and graduate the arc XY in degrees. Through these points of division (P), produce the

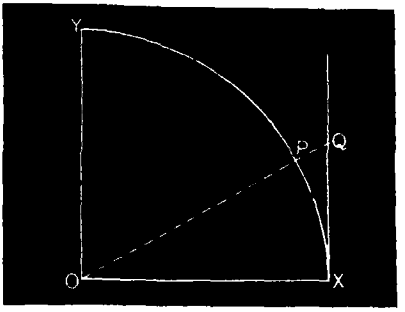


FIG. 25.

radius (OP) to meet the tangent at X in a point (Q). Measure the distance of each of these points from X, true to ·01 in., divide by 10 and thus construct a **table of tangents** true to three places of decimals.

Compare the results with the tables on pp. 38, 39.

It will be found that as the angle XOP approaches 90° the value of the tangent is very large and increases as the angle increases. The student should verify, with the values obtained for angles between 45° and 60° and 45° and 30°, that the formula

$$\tan A \times \tan (90° - A) = 1$$

is true; and, *assuming* this universally true between 0° and 90° (cf. p. 28), obtain the tangents for angles between 60° and 90° from the values observed for those between 30° and 0°. [Use decimal approximation to three places.]

1. Shew that $\tan A = \sin A \div \cos A$.

2. Let ABC be a triangle, right-angled at C; produce BC to D, making $CD = BC$; join AD, and draw BE perpendicular to AD.

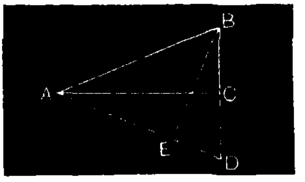


FIG. 26.

Take $AB = 1$, and $\angle BAC = A$; shew that

$$BE = \sin 2A$$
$$BD = 2 \sin A,$$

and hence, in general, when A is an acute angle less than 45°, that

$$\sin 2A \neq 2 \sin A.$$

3. Let ABC be a triangle, right-angled at C; produce AC to D, making $BD = AC$; join BD, and draw DE perpendicular to AB produced, meeting it in E.

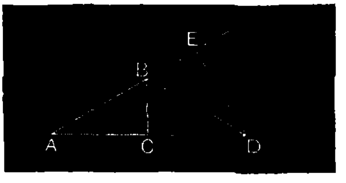


FIG. 27.

Take $AB = 1$, and $\angle BAC = A$; shew that

$$BE = \cos 2A,$$
$$AD = 2 \cos A,$$

and hence, when A is any acute angle less than 45°, that

$$\cos 2A \neq 2 \cos A.$$

4. Let ABC be a triangle, right-angled at C: make $AD = AC$ and $BD = BC$, produce DB and AC to meet in E.

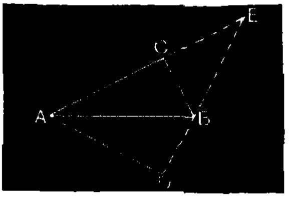


FIG. 28.

Take $AC = 1$ and $\angle BAC = A$, and shew, when A is *any acute* angle less than 45°, that

$$\tan 2A \neq 2 \tan A.$$

In Expts. **21, 22** the **sines, cosines** and **tangents** of given angles have been found: if the ratios are given the angles can be found by similar methods.

23. Find an angle whose sine is $\frac{m}{n}$.

Draw a quadrant of a circle with a convenient radius whose length represents n units on some chosen scale. Draw OX,

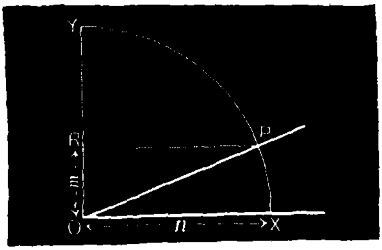


FIG. 29.

OY two radii at right angles. cut off OR along OY so that OR$=m$ units on the chosen scale. Draw RP $\parallel$ OX to cut the arc in P.

Then $\angle$ XOP is an angle whose sine is $\frac{m}{n}$. [Why?]

An alternative construction may be verified:

Take OR$=m$ units and draw RP perpendicular to OR. With centre O and radius$=n$ units describe an arc cutting R in P.

Then

$$\sin \text{RPO} = \frac{m}{n}.$$

FIG. 30.

24. Find an angle whose cosine is $\frac{m}{n}$.

25. Find an angle whose tangent is $\frac{m}{n}$.

The notation most commonly used for denoting angles whose sines, cosines or tangents are given quantities is the following:

$$\sin^{-1}\mathbf{x} \equiv \text{an angle whose sine is } \mathbf{x},$$
$$\cos^{-1}\mathbf{y} \equiv \text{an angle whose cosine is } \mathbf{y},$$
$$\tan^{-1}\mathbf{z} \equiv \text{an angle whose tangent is } \mathbf{z}.$$

It must be carefully observed that although the "-1" occupies the position of an index, yet $\sin^{-1}x$ must not be confused with $(\sin x)^{-1}$, i.e. $1 \div \sin x$. This is all the more important because in all other cases the powers of trigonometrical ratios are indicated by attaching the index to the name of the ratio.

Thus
$$\sin^2 A \equiv (\sin A)^2,$$
$$\tan^{\frac{1}{2}} 2A \equiv \sqrt{\tan 2A}.$$

This exceptional use of the symbol $^{-1}$, however, need not lead to any confusion; for the reciprocals of the sine, cosine and tangent have, as will be seen in § 8, special names given to them, so that the expressions

$$(\sin x)^{-1},\ (\cos x)^{-1},\ (\tan x)^{-1}$$

are not used.

Note. The continental notation for $\sin^{-1} x$, $\cos^{-1} x$, etc. is arc-sin x, arc-cos x, etc.

It should also be noticed that in the expression $\sin^{-1} x$, the symbol x stands for a *number*; in sin A, the symbol A stands for *an angle*.

1. With scale and compasses make angles

 (i) whose sines are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$;

 (ii) whose cosines are $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$;

 (iii) whose tangents are $\frac{1}{2}$, 2, $\frac{2}{3}$, $\frac{3}{2}$.

2. Find the number of degrees in each of the above angles by means of the tables constructed in Expts. 21, 22. Verify with a protractor or scale of chords.

§ 7. Advantages and use of tables.

When solving problems in "Heights and Distances" the degree of accuracy attainable by the method of drawing to scale, given in § 5, is very limited. Measurements to $\frac{1}{100}$ in. in the length of a line, and to $\frac{1}{2}°$ in the magnitude of an angle, are only to be trusted when using extremely accurate instruments and with most careful drawing. Thus when finding the height of a hill of, say, 4000 ft. by the observed depressions of two consecutive milestones, suppose a length of about 5 in. is taken to represent a mile, according to which the height of the hill would be represented by less than 4 in.; then an error in drawing of ·01 in. would give an error in the height of the hill

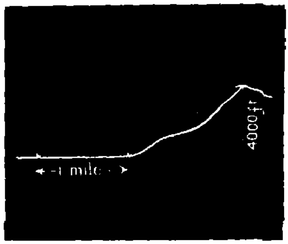


Fig. 31.

$$= \frac{\cdot 01}{4} \times 4000 \text{ ft.}$$

$$= 10 \text{ ft.}$$

Again, very rarely would the landmarks be so conveniently placed that the depressions could be expressed exactly in half-degrees. A much greater error will in general be introduced by inaccuracy in drawing these angles (Expt. **26**) of depression or elevation as the case may be.

26. From the top of a hill the angles of depression of two landmarks known to be 120 yds. apart are 19° and 25° respectively. (Cf. Expt. 18.)

Using a large sheet of paper, take 3″ to represent 120 yds., and find the several heights of the hill given by drawing diagrams

(i) in which an error of half a degree is made in drawing the angle 19° (2 figures);

(ii) in which an error of half a degree is made in drawing the angle 25° (2 figures);

(iii) in which an error of half a degree is made in drawing both angles (4 figures).

27. Calculate the height of the hill, using the values of the tangents found in Expt. 22, as follows.

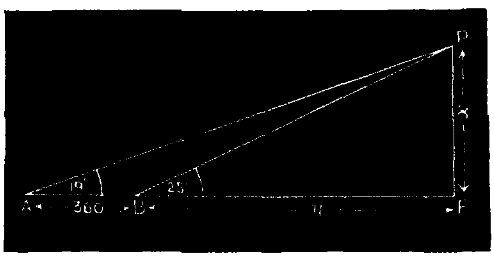


FIG. 32.

Let A and B be the landmarks 120 yds. apart, P the top of the hill, F the foot of the vertical through P; let PF $=x$ feet, BF $=y$ feet.

Then $$AF = y + 360;$$

$$\therefore \quad \frac{x}{y+360} = \tan 19°$$

$$\text{and} \quad \frac{x}{y} = \tan 25°;$$

$$\therefore \quad \frac{y+360}{x} - \frac{y}{x} = \frac{1}{\tan 19°} - \frac{1}{\tan 25};$$

$$\therefore \frac{y+360}{x} - \frac{y}{x} = \frac{360}{x} = \tan 71^\circ - \tan 65^\circ, \qquad \text{[see p. 28]}$$

$$= 2{\cdot}904 - 2{\cdot}145,$$

$$= {\cdot}759;$$

$$\therefore x = \frac{360}{{\cdot}759}$$

$$= 474{\cdot}3 \text{ feet.}$$

The values used are practically extracts from a table of tangents correct to *three decimal places.* By methods given in more advanced Trigonometry, tables can be calculated to any required number of decimal places. Four- and five-figure tables are published for general use in Physics and by engineers; surveyors use seven-figure tables, and for some of the most exact astronomical calculations ten- and even twelve-figure tables are used.

28. Calculate the height of the hill in Expt. **27**, using *five-figure tables.*

[We find from the tables on pp. 38, 39,

$$\tan 71^\circ = 2{\cdot}90421,$$

$$\tan 65^\circ = 2{\cdot}14451,$$

and hence $x = 473{\cdot}87$ ft.]

This result is the true height to a hundredth of an inch, the result obtained by use of seven-figure tables being 473·8688 ft.

In general, four-, five- and seven-figure tables give results in problems such as the above which are correct to four, five and seven significant figures respectively.

Sines	0′	10′	20′	30′	40′	50′	60′	
0°	·00000	·00291	·00582	·00873	·01164	·01454	·01745	89°
1	·01745	·02036	·02327	·02618	·02908	·03199	·03490	88
2	·03490	·03781	·04071	·04362	·04653	·04943	·05234	87
3	·05234	·05524	·05814	·06105	·06395	·06685	·06976	86
4	·06976	·07266	·07556	·07846	·08136	·08426	·08716	85
5	·08716	·09005	·09295	·09585	·09874	·10164	·10453	84
6	·10453	·10742	·11031	·11320	·11609	·11898	·12187	83
7	·12187	·12476	·12764	·13053	·13341	·13629	·13917	82
8	·13917	·14205	·14493	·14781	·15069	·15356	·15643	81
9	·15643	·15931	·16218	·16505	·16792	·17078	·17365	80
10	·17365	·17651	·17937	·18224	·18509	·18795	·19081	79
11	·19081	·19366	·19652	·19937	·20222	·20507	·20791	78
12	·20791	·21076	·21360	·21644	·21928	·22212	·22495	77
13	·22495	·22778	·23062	·23345	·23627	·23910	·24192	76
14	·24192	·24474	·24756	·25038	·25320	·25601	·25882	75
15	·25882	·26163	·26443	·26724	·27004	·27284	·27564	74
16	·27564	·27843	·28123	·28402	·28680	·28959	·29237	73
17	·29237	·29515	·29793	·30071	·30348	·30625	·30902	72
18	·30902	·31178	·31454	·31730	·32006	·32282	·32557	71
19	·32557	·32832	·33106	·33381	·33655	·33929	·34202	70
20	·34202	·34475	·34748	·35021	·35293	·35565	·35837	69
21	·35837	·36108	·36379	·36650	·36921	·37191	·37461	68
22	·37461	·37730	·37999	·38268	·38537	·38805	·39073	67
23	·39073	·39341	·39608	·39875	·40142	·40408	·40674	66
24	·40674	·40939	·41204	·41469	·41734	·41998	·42262	65
25	·42262	·42525	·42788	·43051	·43313	·43575	·43837	64
26	·43837	·44098	·44359	·44620	·44880	·45140	·45399	63
27	·45399	·45658	·45917	·46175	·46433	·46690	·46947	62
28	·46947	·47204	·47460	·47716	·47971	·48226	·48481	61
29	·48481	·48735	·48989	·49242	·49495	·49748	·50000	60
30	·50000	·50252	·50503	·50754	·51004	·51254	·51504	59
31	·51504	·51753	·52002	·52250	·52498	·52745	·52992	58
32	·52992	·53238	·53484	·53730	·53975	·54220	·54464	57
33	·54464	·54708	·54951	·55194	·55436	·55678	·55919	56
34	·55919	·56160	·56401	·56641	·56880	·57119	·57358	55
35	·57358	·57596	·57833	·58070	·58307	·58543	·58779	54
36	·58779	·59014	·59248	·59482	·59716	·59949	·60182	53
37	·60182	·60414	·60645	·60876	·61107	·61337	·61567	52
38	·61567	·61795	·62024	·62251	·62479	·62706	·62932	51
39	·62932	·63158	·63383	·63608	·63832	·64056	·64279	50
40	·64279	·64501	·64723	·64945	·65166	·65386	·65606	49
41	·65606	·65825	·66044	·66262	·66480	·66697	·66913	48
42	·66913	·67129	·67344	·67559	·67773	·67987	·68200	47
43	·68200	·68412	·68624	·68835	·69046	·69256	·69466	46
44	·69466	·69675	·69883	·70091	·70298	·70505	·70711	45
	60′	50′	40′	30′	20′	10′	0′	Co-sines

Sines	0′	10′	20′	30′	40′	50′	60′	
45°	·70711	·70916	·71121	·71325	·71529	·71732	·71934	**44°**
46	·71934	·72136	·72337	·72537	·72737	·72837	·73135	**43**
47	·73135	·73333	·73531	·73728	·73924	·74120	·74314	**42**
48	·74314	·74509	·74703	·74896	·75088	·75280	·75471	**41**
49	·75471	·75661	·75851	·76041	·76229	·76417	·76604	**40**
50	·76604	·76791	·76977	·77162	·77347	·77531	·77715	**39**
51	·77715	·77897	·78079	·78261	·78442	·78622	·78801	**38**
52	·78801	·78980	·79158	·79335	·79512	·79688	·79864	**37**
53	·79864	·80038	·80212	·80386	·80558	·80730	·80902	**36**
54	·80902	·81072	·81242	·81412	·81580	·81748	·81915	**35**
55	·81915	·82082	·82248	·82413	·82577	·82741	·82904	**34**
56	·82904	·83066	·83228	·83389	·83549	·83708	·83867	**33**
57	·83867	·84025	·84182	·84339	·84495	·84650	·84805	**32**
58	·84805	·84959	·85112	·85264	·85416	·85567	·85717	**31**
59	·85717	·85866	·86015	·86163	·86310	·86457	·86603	**30**
60	·86603	·86748	·86892	·87036	·87178	·87321	·87462	**29**
61	·87462	·87603	·87743	·87882	·88020	·88158	·88295	**28**
62	·88295	·88431	·88566	·88701	·88835	·88968	·89101	**27**
63	·89101	·89232	·89363	·89493	·89623	·89751	·89879	**26**
64	·89879	·90007	·90133	·90259	·90383	·90507	·90631	**25**
65	·90631	·90753	·90875	·90996	·91116	·91236	·91355	**24**
66	·91355	·91472	·91590	·91706	·91822	·91936	·92050	**23**
67	·92050	·92164	·92276	·92388	·92499	·92609	·92718	**22**
68	·92718	·92827	·92935	·93042	·93148	·93253	·93358	**21**
69	·93358	·93462	·93565	·93667	·93769	·93869	·93969	**20**
70	·93969	·94068	·94167	·94264	·94361	·94457	·94552	**19**
71	·94552	·94646	·94740	·94832	·94924	·95015	·95106	**18**
72	·95106	·95195	·95284	·95372	·95459	·95545	·95630	**17**
73	·95630	·95715	·95799	·95882	·95964	·96046	·96126	**16**
74	·96126	·96206	·96285	·96363	·96440	·96517	·96593	**15**
75	·96593	·96667	·96742	·96815	·96887	·96959	·97030	**14**
76	·97030	·97100	·97169	·97237	·97304	·97371	·97437	**13**
77	·97437	·97502	·97566	·97630	·97692	·97754	·97815	**12**
78	·97815	·97875	·97934	·97992	·98050	·98107	·98163	**11**
79	·98163	·99218	·98272	·98325	·98378	·98430	·98481	**10**
80	·98481	·98531	·98580	·98629	·98676	·98723	·98769	**9**
81	·98769	·98814	·98858	·98902	·98944	·98986	·99027	**8**
82	·99027	·99067	·99106	·99144	·99182	·99219	·99255	**7**
83	·99255	·99290	·99324	·99357	·99390	·99421	·99452	**6**
84	·99452	·99482	·99511	·99540	·99567	·99594	·99619	**5**
85	·99619	·99644	·99668	·99692	·99714	·99736	·99756	**4**
86	·99756	·99776	·99795	·99813	·99831	·99847	·99863	**3**
87	·99863	·99878	·99892	·99905	·99917	·99929	·99939	**2**
88	·99939	·99949	·99958	·99966	·99973	·99979	·99985	**1**
89	·99985	·99989	·99993	·99996	·99998	·99999	1·00000	**0**
	60′	**50′**	**40′**	**30′**	**20′**	**10′**	**0′**	Co-sines

Tangents	0′	10′	20′	30′	40′	50′	60′	
0°	0·00000	0·00291	0·00582	0·00873	0·01164	0·01455	0·01746	89°
1	0·01746	0·02037	0·02328	0·02619	0·02910	0·03201	0·03492	88
2	0·03492	0·03783	0·04075	0·04366	0·04658	0·04949	0·05241	87
3	0·05241	0·05533	0·05824	0·06116	0·06408	0·06700	0·06993	86
4	0·06993	0·07285	0·07578	0·07870	0·08163	0·08456	0·08749	85
5	0·08749	0·09042	0·09335	0·09629	0·09923	0·10216	0·10510	84
6	0·10510	0·10805	0·11099	0·11394	0·11688	0·11983	0·12278	83
7	0·12278	0·12574	0·12869	0·13165	0·13461	0·13758	0·14054	82
8	0·14054	0·14351	0·14648	0·14945	0·15243	0·15540	0·15838	81
9	0·15838	0·16137	0·16435	0·16734	0·17033	0·17333	0·17633	80
10	0·17633	0·17933	0·18233	0·18534	0·18835	0·19136	0·19438	79
11	0·19438	0·19740	0·20042	0·20345	0·20648	0·20952	0·21256	78
12	0·21256	0·21560	0·21864	0·22169	0·22475	0·22781	0·23087	77
13	0·23087	0·23393	0·23700	0·24008	0·24316	0·24624	0·24933	76
14	0·24933	0·25242	0·25552	0·25862	0·26172	0·26483	0·26795	75
15	0·26795	0·27107	0·27419	0·27732	0·28046	0·28360	0·28675	74
16	0·28675	0·28990	0·29305	0·29621	0·29938	0·30255	0·30573	73
17	0·30573	0·30891	0·31210	0·31530	0·31850	0·32171	0·32492	72
18	0·32492	0·32814	0·33136	0·33460	0·33783	0·34108	0·34433	71
19	0·34433	0·34758	0·35085	0·35412	0·35740	0·36068	0·36397	70
20	0·36397	0·36727	0·37057	0·37388	0·37720	0·38053	0·38386	69
21	0·38386	0·38721	0·39055	0·39391	0·39727	0·40065	0·40403	68
22	0·40403	0·40741	0·41081	0·41421	0·41763	0·42105	0·42447	67
23	0·42447	0·42791	0·43136	0·43481	0·43828	0·44175	0·44523	66
24	0·44523	0·44872	0·45222	0·45573	0·45924	0·46277	0·46631	65
25	0·46631	0·46985	0·47341	0·47698	0·48055	0·48414	0·48773	64
26	0·48773	0·49134	0·49495	0·49858	0·50222	0·50587	0·50953	63
27	0·50953	0·51320	0·51688	0·52057	0·52427	0·52798	0·53171	62
28	0·53171	0·53545	0·53920	0·54296	0·54673	0·55051	0·55431	61
29	0·55431	0·55812	0·56194	0·56577	0·56962	0·57348	0·57735	60
30	0·57735	0·58124	0·58513	0·58905	0·59297	0·59691	0·60086	59
31	0·60086	0·60483	0·60881	0·61280	0·61681	0·62083	0·62487	58
32	0·62487	0·62892	0·63299	0·63707	0·64117	0·64528	0·64941	57
33	0·64941	0·65355	0·65771	0·66189	0·66608	0·67028	0·67451	56
34	0·67451	0·67875	0·68301	0·68728	0·69157	0·69588	0·70021	55
35	0·70021	0·70455	0·70891	0·71329	0·71769	0·72211	0·72654	54
36	0·72654	0·73100	0·73547	0·73996	0·74447	0·74900	0·75355	53
37	0·75355	0·75812	0·76272	0·76733	0·77196	0·77661	0·78129	52
38	0·78129	0·78598	0·79070	0·79544	0·80020	0·80498	0·80978	51
39	0·80978	0·81461	0·81946	0·82434	0·82923	0·83415	0·83910	50
40	0·83910	0·84407	0·84906	0·85408	0·85912	0·86419	0·86929	49
41	0·86929	0·87441	0·87955	0·88473	0·88992	0·89515	0·90040	48
42	0·90040	0·90569	0·91099	0·91633	0·92170	0·92709	0·93252	47
43	0·93252	0·93797	0·94345	0·94896	0·95451	0·96008	0·96569	46
44	0·96569	0·97133	0·97700	0·98270	0·98843	0·99420	1·00000	45
	60′	50′	40′	30′	20′	10′	0′	Cotangents

0′	10′	20′	30′	40′	50′	60′	
1·00000	1·00583	1·01770	1·01761	1·02355	1·02952	1·03553	44°
1·03553	1·04158	1·04766	1·05378	1·05994	1·06613	1·07237	43
1·07237	1·07864	1·08496	1·09131	1·09770	1·10414	1·11061	42
1·11061	1·11713	1·12369	1·13029	1·13694	1·14363	1·15037	41
1·15037	1·15715	1·16398	1·17085	1·17777	1·18474	1·19175	40
1·19175	1·19882	1·20593	1·21310	1·22031	1·22758	1·23490	39
1·23490	1·24227	1·24969	1·25717	1·26471	1·27230	1·27994	38
1·27994	1·28764	1·29541	1·30322	1·31110	1·31904	1·32704	37
1·32704	1·33511	1·34323	1·35142	1·35968	1·36800	1·37638	36
1·37638	1·38484	1·39336	1·40195	1·41061	1·41934	1·42815	35
1·42815	1·43703	1·44598	1·45501	1·46411	1·47330	1·48256	34
1·48256	1·49190	1·50133	1·51084	1·52043	1·53010	1·53987	33
1·53987	1·54972	1·55966	1·56969	1·57981	1·59002	1·60033	32
1·60033	1·61074	1·62125	1·63185	1·64256	1·65337	1·66428	31
1·66428	1·67530	1·68643	1·69766	1·70901	1·72047	1·73205	30
1·73205	1·74375	1·75556	1·76749	1·77955	1·79174	1·80405	29
1·80405	1·81649	1·82906	1·84177	1·85462	1·86760	1·88073	28
1·88073	1·89400	1·90741	1·92098	1·93470	1·94858	1·96261	27
1·96261	1·97681	1·99116	2·00569	2·02039	2·03526	2·05030	26
2·05030	2·06553	2·08094	2·09654	2·11233	2·12832	2·14451	25
2·14451	2·16090	2·17749	2·19430	2·21132	2·22857	2·24604	24
2·24604	2·26374	2·28167	2·29984	2·31826	2·33693	2·35585	23
2·35585	2·37504	2·39449	2·41421	2·43422	2·45451	2·47509	22
2·47509	2·49597	2·51715	2·53865	2·56046	2·58261	2·60509	21
2·60509	2·62791	2·65109	2·67462	2·69853	2·72281	2·74748	20
2·74748	2·77254	2·79802	2·82391	2·85023	2·87700	2·90421	19
2·90421	2·93189	2·96004	2·98869	3·01783	3·04749	3·07768	18
3·07768	3·10842	3·13972	3·17159	3·20406	3·23714	3·27085	17
3·27085	3·30521	3·34023	3·37594	3·41236	3·44951	3·48741	16
3·48741	3·52609	3·56557	3·60588	3·64705	3·68909	3·73205	15
3·73205	3·77595	3·82083	3·86671	3·91364	3·96165	4·01078	14
4·01078	4·06107	4·11256	4·16530	4·21933	4·27471	4·33148	13
4·33148	4·38969	4·44942	4·51071	4·57363	4·63825	4·70463	12
4·70463	4·77286	4·84300	4·91516	4·98940	5·06584	5·14455	11
5·14455	5·22566	5·30928	5·39552	5·48451	5·57638	5·67128	10
5·67128	5·76937	5·87080	5·97576	6·08444	6·19703	6·31375	9
6·31375	6·43484	6·56055	6·69116	6·82694	6·96823	7·11537	8
7·11537	7·26873	7·42871	7·59575	7·77035	7·95302	8·14435	7
8·14435	8·34496	8·55555	8·77689	9·00983	9·25530	9·51436	6
9·51436	9·78817	10·07803	10·38540	10·71191	11·05943	11·43005	5
11·43005	11·82617	12·25051	12·70620	13·19688	13·72674	14·30067	4
14·30067	14·92442	15·60478	16·34986	17·16934	18·07498	19·08114	3
19·08114	20·20555	21·47040	22·90377	24·54176	26·43160	28·63625	2
28·63625	31·24158	34·36777	38·18846	42·96408	49·10388	57·28996	1
57·28996	68·75009	85·93979	114·58865	171·88540	343·77371	∞	0
60′	50′	40′	30′	20′	10′	0′	Cotangents

Secants	0′	10′	20′	30′	40′	50′	60′	
0°	1·00000	1·00000(4)	1·00001	1·00004	1·00007	1·00011	1·00015	89°
1	1·00015	1·00021	1·00027	1·00034	1·00042	1·00051	1·00061	88
2	1·00061	1·00072	1·00083	1·00095	1·00108	1·00122	1·00137	87
3	1·00137	1·00153	1·00169	1·00187	1·00205	1·00224	1·00244	86
4	1·00244	1·00265	1·00287	1·00309	1·00333	1·00357	1·00382	85
5	1·00382	1·00408	1·00435	1·00463	1·00491	1·00521	1·00551	84
6	1·00551	1·00582	1·00614	1·00647	1·00681	1·00715	1·00751	83
7	1·00751	1·00787	1·00825	1·00863	1·00902	1·00942	1·00983	82
8	1·00983	1·01024	1·01067	1·01111	1·01155	1·01200	1·01247	81
9	1·01247	1·01294	1·01342	1·01391	1·01440	1·01491	1·01543	80
10	1·01543	1·01595	1·01649	1·01703	1·01758	1·01815	1·01872	79
11	1·01872	1·01930	1·01989	1·02049	1·02110	1·02171	1·02234	78
12	1·02234	1·02298	1·02362	1·02428	1·02494	1·02562	1·02630	77
13	1·02630	1·02700	1·02770	1·02842	1·02914	1·02987	1·03061	76
14	1·03061	1·03137	1·03213	1·03290	1·03368	1·03447	1·03528	75
15	1·03528	1·03609	1·03691	1·03774	1·03858	1·03944	1·04030	74
16	1·04030	1·04117	1·04206	1·04295	1·04385	1·04477	1·04569	73
17	1·04569	1·04663	1·04757	1·04853	1·04950	1·05047	1·05146	72
18	1·05146	1·05246	1·05347	1·05449	1·05552	1·05657	1·05762	71
19	1·05762	1·05869	1·05976	1·06085	1·06195	1·06306	1·06418	70
20	1·06418	1·06531	1·06645	1·06761	1·06878	1·06995	1·07115	69
21	1·07115	1·07235	1·07356	1·07479	1·07602	1·07727	1·07853	68
22	1·07853	1·07981	1·08109	1·08239	1·08370	1·08503	1·08636	67
23	1·08636	1·08771	1·08907	1·09044	1·09183	1·09323	1·09464	66
24	1·09464	1·09606	1·09750	1·09895	1·10041	1·10189	1·10338	65
25	1·10338	1·10488	1·10640	1·10793	1·10947	1·11103	1·11260	64
26	1·11260	1·11419	1·11579	1·11740	1·11903	1·12067	1·12233	63
27	1·12233	1·12400	1·12568	1·12738	1 12910	1·13083	1·13257	62
28	1·13257	1·13433	1·13610	1·13789	1·13970	1·14152	1·14335	61
29	1·14335	1·14521	1·14707	1·14896	1·15085	1·15277	1·15470	60
30	1·15470	1·15665	1·15861	1·16059	1·16259	1·16460	1·16663	59
31	1·16663	1·16868	1·17075	1·17283	1·17493	1·17704	1·17918	58
32	1·17918	1·18133	1·18350	1·18569	1·18790	1·19012	1·19236	57
33	1·19236	1·19463	1·19691	1·19920	1·20152	1·20386	1·20622	56
34	1·20622	1·20859	1·21099	1·21341	1·21584	1·21830	1·22077	55
35	1·22077	1·22327	1·22579	1·22833	1·23089	1·23347	1·23607	54
36	1·23607	1·23869	1·24136	1·24400	1·24669	1·24940	1·25214	53
37	1·25214	1·25489	1·25767	1·26047	1·26330	1·26615	1·26902	52
38	1·26902	1·27191	1·27483	1·27778	1·28075	1·28374	1·28676	51
39	1·28676	1·28980	1·29287	1·29597	1·29909	1·30223	1·30541	50
40	1·30541	1·30861	1·31183	1·31509	1·31837	1·32168	1·32501	49
41	1·32501	1·32838	1·33177	1·33519	1·33864	1·34212	1·34563	48
42	1·34563	1·34917	1·35274	1·35634	1·35997	1·36363	1·36733	47
43	1·36733	1·37105	1·37481	1·37860	1·38242	1·38628	1·39016	46
44	1·39016	1·39409	1·39804	1·40203	1·40606	1·41012	1·41421	45
	60′	50′	40′	·30′	20′	10′	0′	Cosecants

	0′	10′	20′	30′	40′	50′	60′	
5°	1·41421	1·41835	1·42251	1·42672	1·43096	1·43524	1·43956	44°
6	1·43956	1·44391	1·44831	1·45274	1·45721	1·46173	1·46628	43
7	1·46628	1·47087	1·47551	1·48019	1·48491	1·48967	1·49448	42
8	1·49448	1·49933	1·50422	1·50916	1·51415	1·51918	1·52425	41
9	1·52425	1·52938	1·53455	1·53977	1·54504	1·55036	1·55572	40
0	1·55572	1·56114	1·56661	1·57213	1·57771	1·58333	1·58902	39
1	1·58902	1·59475	1·60054	1·60639	1·61229	1·61825	1·62427	38
2	1·62427	1·63035	1·63648	1·64268	1·64894	1·65526	1·66164	37
3	1·66164	1·66809	1·67460	1·68117	1·68782	1·69452	1·70130	36
4	1·70130	1·70815	1·71506	1·72205	1·72911	1·73624	1·74345	35
5	1·74345	1·75073	1·75808	1·76552	1·77303	1·78062	1·78829	34
6	1·78829	1·79604	1·80388	1·81180	1·81981	1·82790	1·83608	33
7	1·83608	1·84435	1·85271	1·86116	1·86990	1·87834	1·88708	32
8	1·88708	1·89591	1·90485	1·91388	1·92302	1·93226	1·94160	31
9	1·94160	1·95106	1·96062	1·97029	1·98008	1·98998	2·00000	30
60	2·00000	2·01014	2·02039	2·03077	2·04128	2·05191	2·06267	29
61	2·06267	2·07356	2·08458	2·09574	2·10704	2·11847	2·13005	28
62	2·13005	2·14178	2·15366	2·16568	2·17786	2·19019	2·20269	27
63	2·20269	2·21535	2·22817	2·24116	2·25432	2·26766	2·28117	26
64	2·28117	2·29487	2·30875	2·32282	2·33708	2·35154	2·36620	25
65	2·36620	2·38107	2·39614	2·41142	2·42692	2·44264	2·45859	24
66	2·45859	2·47477	2·49119	2·50784	2·52474	2·54190	2·55930	23
67	2·55930	2·57698	2·59491	2·61313	2·63162	2·65040	2·66947	22
68	2·66947	2·68884	2·70851	2·72850	2·74881	2·76945	2·79043	21
69	2·79043	2·81175	2·83342	2·85545	2·87785	2·90063	2·92380	20
70	2·92380	2·94737	2·97135	2·99574	3·02057	3·04584	3·07155	19
71	3·07155	3·09774	3·12440	3·15155	3·17920	3·20737	3·23607	18
72	3·23607	3·26531	3·29512	3·32551	3·35649	3·38808	3·42030	17
73	3·42030	3·45317	3·48671	3·52094	3·55587	3·59154	3·62796	16
74	3·62796	3·66515	3·70315	3·74198	3·78166	3·82223	3·86370	15
75	3·86370	3·90613	3·94952	3·99393	4·03938	4·08591	4·13357	14
76	4·13357	4·18238	4·23239	4·28366	4·33622	4·39012	4·44541	13
77	4·44541	4·50216	4·56041	4·62023	4·68167	4·74482	4·80973	12
78	4·80973	4·87649	4·94517	5·01585	5·08863	5·16359	5·24084	11
79	5·24084	5·32049	5·40263	5·48740	5·57493	5·66533	5·75877	10
80	5·75877	5·85539	5·95536	6·05886	6·16607	6·27719	6·39245	9
81	6·39245	6·51208	6·63633	6·76547	6·89979	7·03962	7·18530	8
82	7·18530	7·33719	7·49571	7·66130	7·83443	8·01565	8·20551	7
83	8·20551	8·40466	8·61379	8·83367	9·06515	9·30917	9·56677	6
84	9·56677	9·83912	10·12752	10·43343	10·75849	11·10455	11·47371	5
85	11·47371	11·86837	12·29125	12·74550	13·23472	13·76312	14·33559	4
86	14·33559	14·95788	15·63739	16·38041	17·19843	28·10262	19·10732	3
87	19·10732	20·23028	21·49367	22·92559	24·56212	26·45051	28·65371	2
88	28·65371	31·25758	34·38232	38·20155	42·97571	49·11406	57·29869	1
89	57·29869	68·75736	85·94561	114·59301	171·88831	343·77516	∞	0
	60′	50′	40′	30′	20′	10′	0′	Cosecants

EXERCISES.

[*The following exercises are to be worked with the tables on pp.* 36—41. *The answers should be* roughly verified *by a figure drawn to scale.*]

1. Two adjacent sides of a parallelogram are of length 15 and 24 units respectively, and the angle between them is 60°; find the lengths of both diagonals.

2. A rectangular garden, 30 yds. long and 25 yds. wide, is watered by a hose, which will deliver water to a distance of 22 feet from the nozzle. What length of hose is required to reach every part of the garden, if it is attached to a stand-pipe in the middle of one of the longer sides of the garden?

3. A dog-kennel is to have its roof covered with felt. The kennel is 6 feet long, 4 feet broad and its height is 5 feet to the ridge and 3 feet 9 inches to the eaves. How many square feet of felt are required?

4. A wooden foot-bridge is thrown across a railway cutting, 30 feet in depth; the width of the railroad is 27 feet and the banks have a slope of 35°: find the length of the bridge, allowing 6 feet overlap at each end.

5. A person, whose eyes are 5 ft. 6 in. above the ground, is standing within a railway tunnel whose height is 25 feet. The elevations of the tops of the arches at its ends are 5° 37′, 13° 22′ respectively. Find the length of the tunnel.

6. Two persons, at stations 1000 yds. apart, due East and West, simultaneously observe a balloon. At the westerly station the balloon bears N.E. by E. and its elevation is 47°; at the other station it bears N.W. by N.: shew that its elevation is 58° 4′ at this station, and find the height of the balloon.

7. A church stands in the centre of a square, the summit of the spire being vertically over the middle point of the square. When the sun has an altitude of 33° 24′, the shadow of the spire just reaches a corner of the square. If the square contains $2\frac{1}{2}$ acres find the height of the spire.

8. A ring 10 inches in diameter is suspended by six equal strings, attached to its circumference at equal intervals, from a point 1 foot above its centre: find the angle between two consecutive strings.

9. The guns from a fort on the top of a cliff will carry a distance of $1\frac{1}{2}$ miles; they can be depressed 15° for point-blank range: what is the breadth of the danger-zone, if the height of the cliff is 250 feet?

10. A gun, pointing through an embrasure of a fort can swing 15° to either side. A ship steaming at 22 knots an hour passes the fort at a least distance of one knot: how long is she under fire from the gun?

11. A man, on the top of a hill, observes the angles of depression of the top and bottom of a tower 50 feet high at the foot of the hill to be $5^\circ\ 49'$, $6'\ 2''$ respectively: find the height of the hill.

12. A tower stands on the side of a hill, rising 1 in 25 uniformly. After proceeding 200 feet straight up the hill the angle of elevation of the top of the tower is $34^\circ\ 22'$. Find the height of the tower.

§ 8. Connections between the trigonometrical ratios of a given angle.

Besides the sine, cosine and tangent of an angle other trigonometrical ratios are used.

Those in general use are the reciprocals of the sine, cosine and tangent. These are called respectively the **cosecant, secant** and **cotangent**; and the abbreviations cosec., sec., tan. are used.

Thus we have, in any right-angled triangle,

$$\text{cosec}(\text{angle}) = \frac{1}{\sin(\text{angle})} = \frac{\text{side opposite right angle}}{\text{side opposite angle}},$$

$$\sec(\text{angle}) = \frac{1}{\cos(\text{angle})} = \frac{\text{side opposite right angle}}{\text{side opposite other acute angle}},$$

$$\cot(\text{angle}) = \frac{1}{\tan(\text{angle})} = \frac{\text{side opposite other acute angle}}{\text{side opposite angle}}.$$

Two others are used, chiefly in arch construction, the versed-sine and the co-versed-sine. They are defined as follows:

$$\text{vers}(\text{angle}) = 1 - \cos(\text{angle}),$$

$$\text{covers}(\text{angle}) = 1 - \sin(\text{angle}).$$

The derivation of the names of the ratios is instructive. The earlier mathematicians defined the sine, cosine, etc. as the *measures of certain lines in a circle* whose *radius* was taken as the unit of length.

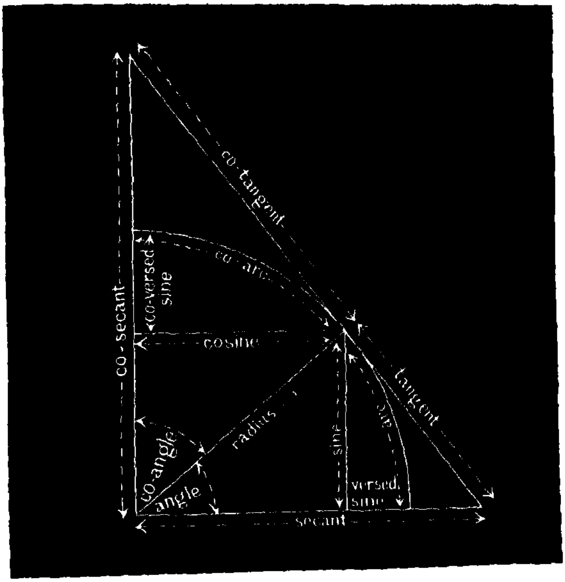


FIG. 33.

Fig. 33 shews these lines: and from it the derivation of "tangent" and "secant" are evident, whilst the word "arc" is the key to the derivation of the word "sine." The curve is looked on as the half of a bow (L. *arcus*) of which the "sine" is half the string by which the bow is bent (L. *sinus*, a bending; cf. sinew, A.S. *sinu*). The line denoting the versed sine was sometimes called the *sagitta* (L. an *arrow*). Originally the length of the whole chord of the double arc was given instead of the sine of the arc. The "versed sine" is the sine turned through a right angle about its foot, whilst its length decreases to allow its other extremity to remain on the circle.

The other four ratios are the ratios of the *complementary* angle, or arc as the early mathematicians considered it (see Circular Measure, § 20).

This notation was found inconvenient as the "unit" used for the radius had always to be stated, and the idea of ratios was introduced into general use by Euler about 1780.

29. Let BAC be any acute angle, P a point in one arm of the angle and PM perpendicular to the other arm.

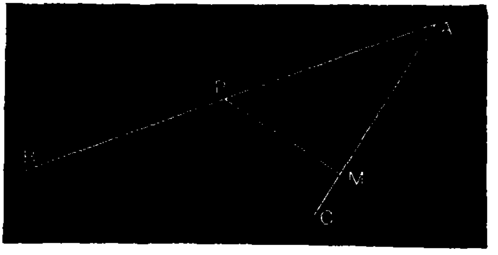


FIG. 34.

Then by Pythagoras' Theorem, in the right-angled triangle AMP, we have

$$AM^2 + MP^2 = AP^2.$$

Dividing each side of this equation by (i) AP^2, (ii) AM^2, (iii) MP^2, prove

$$\text{(i)} \quad \cos^2 A + \sin^2 A = 1,$$
$$\text{(ii)} \quad \sec^2 A - \tan^2 A = 1,$$
$$\text{(iii)} \quad \operatorname{cosec}^2 A - \cot^2 A = 1.$$

These connections between the trigonometrical ratios, together with those derived from the definitions already given, viz.:

$$\text{(iv)} \quad \tan A = \frac{\sin A}{\cos A},$$
$$\text{(v)} \quad \tan A \,.\, \cot A = 1,$$
$$\text{(vi)} \quad \cos A \,.\, \sec A = 1,$$
$$\text{(vii)} \quad \sin A \,.\, \operatorname{cosec} A = 1,$$

enable us to find all the ratios of a given angle if one of them is given.

The values of all the ratios can be expressed in terms of any one ratio. Thus for $\tan A = t$, construct a triangle ABC, right-angled at B, in which $AB = 1$, $BC = t$; then,

since $AC^2 = AB^2 + BC^2$, the values of the other ratios can be at once written down.

For, $\quad \because \; AC = \sqrt{1+t^2}.$

$$\therefore \; \sin\theta = \frac{CB}{CA} = \frac{t}{\sqrt{1+t^2}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$\cos\theta = \frac{AB}{AC} = \frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+\tan^2\theta}}$$

and so on.

The same results may be obtained by the use of the seven formulae on page 46.

For instance,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}.$$

The student should obtain, in this manner, the values of all the ratios in terms of any one ratio and tabulate them. [*See Answers.*]

In any particular example, however, it is often better to proceed from first principles, rather than to quote these relations.

30. Given $\sin P = \frac{60}{61}$, find tan A, cosec A.

Draw a triangle ABC right-angled at C. Let $AC = 60$ units, $AB = 61$ units (the triangle need not be drawn to scale).

Then $\angle ABC = P$. Shew that $BC = 11$ units, and hence

$$\tan A = \frac{60}{11},$$

$$\text{cosec}\, A = \frac{61}{60}.$$

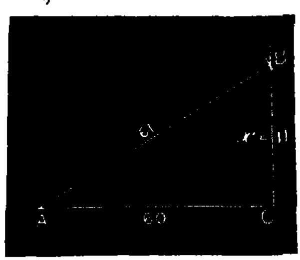


FIG. 35.

EXERCISES.

1. Given that $\sin A = \frac{3}{5}$, find $\tan A$ and $\operatorname{cosec} A$.
2. Given that $\cos B = \frac{1}{3}$, find $\sin B$ and $\cot B$
3. Given that $\tan A = \frac{4}{3}$, find $\sin A$ and $\sec A$
4. Given that $\sec\theta = 4$, find $\cot\theta$ and $\sin\theta$
5. Given that $\tan\theta = \sqrt{3}$, find $\sin\theta$ and $\cos\theta$
6. Given that $\cot\theta = \dfrac{2}{\sqrt{5}}$, find $\sin\theta$ and $\sec\theta$
7. Given that $\sin\theta = \dfrac{b}{c}$, find $\tan\theta$.
8. Given that $\tan\theta = \dfrac{a}{b}$, find $\sin\theta$ and $\cos\theta$
9. Given that $\cos\theta = \dfrac{1}{a}$, find $\sin\theta$ and $\cot\theta$
10. If $\sin\theta = a$, and $\tan\theta = b$, prove that $(1 - a^2)(1 + b^2) = 1$.
11. If $\cos\theta = h$, and $\tan\theta = k$, find the equation connecting h and k.

§ 9. Graphical representation of the trigonometrical ratios.

The tables given on pp. 36–41 for the trigonometrical ratios are too bulky to give a clear mental impression of the variation of the ratios for the whole range between 0° and 90°. If, however, rough approximations to the values there given are used to draw graphs, the diagrams obtained make the variation for the whole range evident at a glance.

31. Draw a graph of $y = \sin x$. Take two straight lines, Ox Oy, at right angles to one another.

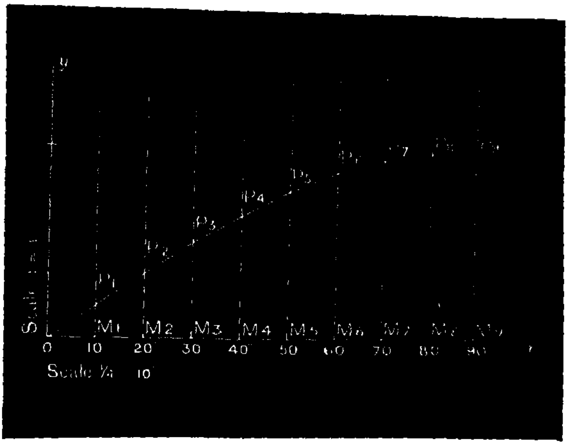


FIG. 36.

Take any convenient unit (say $\frac{1}{2}$ in.) along Ox to represent 10°, and mark off points M_1, M_2, ... M_9, so that OM_1, OM_2, ... OM_9 represent 10°, 20°, ... 90°.

Take 2 in. along Oy to represent unity: then, using either a diagonal scale reading to ·01 in. or squared paper ruled in inches and tenths, the values of the **sines** can be graphically shewn, *true to two places of decimals*, by erecting perpendiculars M_1P_1, M_2P_2, ... M_9P_9 to represent on this scale (2 in. = 1) the values of sin 10°, sin 20°, ... sin 90°, according to the following table, which is taken, true to two places, from the tables on pp. 36, 37. Fig. 36 is drawn half-size.

x (angle) =	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
y (sine) =	0·00	0·17	0·34	0·50	0·64	0·77	0·87	0·94	0·98	1·00

Join OP_1, P_1P_2, ... P_8P_9. Observe that the angles which OP_1, P_1P_2, ... P_8P_9 make with Ox are all different, and steadily decrease.

If, instead of the broken line $OP_1 \ldots P_9$, a curve is drawn freehand, passing with a fair sweep through the points $O, P_1, P_2, \ldots P_9$, a graph for $y = \sin x$ is obtained for *all* values of x between 0° and 90°. Observe that in no place does the curve depart very far from the broken line: thus the **angle of slope** of each of the straight lines $OP_1, P_1P_2, \ldots P_8P_9$ gives very nearly the average angle of slope of the curve for each interval of 10°. Also notice that the agreement between the parts of the curve and the straight lines becomes closer as the angle increases.

If the freehand curve is well drawn on a fairly large scale—the use of "French curves" or "Brooks' flexible curves" are recommended—the value of $\sin x$ can be approximately determined for any acute angle, or the value of any acute angle x whose sine is given. The process is called **graphical interpolation.**

32. **Find sin 39° graphically.**

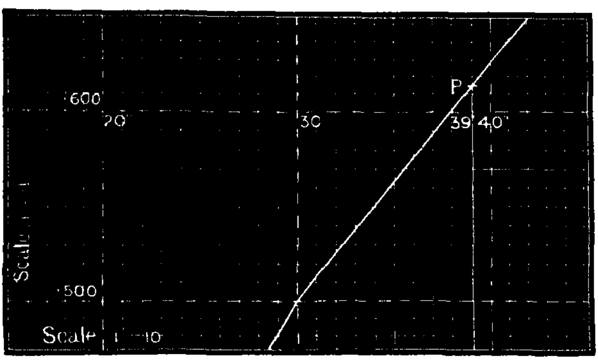


Fig. 37.

Construct a graph of $y = \sin x$ on a large sheet of paper, taking 1 in. along Ox to represent 10°, and 10 in. along or parallel to Oy to represent unity. A part of the graph is shewn in Fig. 37.

Take the point **M** so that **OM** represents 39°, erect a perpendicular **MP** cutting the graph of $y = \sin x$ in **P**, then **MP** represents (on the scale 10 in = 1) the value of $\sin 39°$.

Measure **MP** with a diagonal scale, and compare the value obtained with the value given in the tables on p 36

1 Find graphically the values of

(i) $\sin 25°$, $\sin 84°$, $\sin 15°$, $\sin 22\frac{1}{2}°$,

(ii) $\sin^{-1} 0.309$, $\sin^{-1} 0.809$, $\sin^{-1} 0.951$,

and verify the results by comparison with the tables on pp 36, 37

33 Draw the graph of $y = \cos x$, for all values of x between 0° and 90°, taking from the tables on pp 36, 37, the values of $\cos x$ for $x = 10°, 20°, \ldots 90°$, and using the same scales as in Expt **32** Find by graphical interpolation the values of $\cos 26°$, $\cos 83°$, and the angles $\cos^{-1} 0.970$, $\cos^{-1} 0.470$ Verify the results by comparison with the tables

34. Draw graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$, $y = \operatorname{cosec} x$ for all values of x between 0° and 90°

2. *Verify experimentally for the angles 21°, 47°, 62°, that*

$$1 + \tan^2 x = \sec^2 x,$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x,$$

obtaining the values from the graphs drawn in Expt. **34.**

3 Compare by superposition the general shapes of the graphs of $\sin x$ and $\cos x$, and deduce from them that

$$\sin x = \cos (90° - x)$$

4. Shew, similarly, that

$$(1) \quad \tan x = \cot (90° - x),$$

$$(2) \quad \sec x = \operatorname{cosec} (90° - x)$$

§ 10. "Rate of Increase."

35. Using the same horizontal and vertical scales as in Expt. **32**, mark a series of points whose heights above $\mathsf{O}x$ represent the values of $y = \sin x$, *true to three places of decimals*, for intervals of 1°. Join consecutive points, and observe that this broken line is indistinguishable with the scale used (i.e. error indistinguishable $=$ ·01 in., representing ·001 in the value of the sine) from the continuous curve forming the graph of $y = \sin x$. Also join the points corresponding to 10°, 20°, ... 90°, and shew that now the error in the height above $\mathsf{O}x$ may be as much as ·06 in., i.e. an error in the sine of ·006.

36. Take 5 in. along $\mathsf{O}x$ to represent 1°, and 2 in. along $\mathsf{O}y$ to represent ·01, and mark a series of points between (i) 5° and 6°, (ii) 41° and 42°, (iii) 78° and 79°, whose heights above $\mathsf{O}x$ represent the values of $y = \sin x$, *true to four places of decimals*, for intervals of 10′. Fig. 38 on the opposite page is for the interval 41° and 42°; the straight lines $\mathsf{O}x$ and $\mathsf{O}y$ are 131 in. below AB, and 205 in. to the left of AC, respectively. Join consecutive points and shew that the broken line is indistinguishable from the continuous curve forming the graph of $y = \sin x$, and that the error in the value of the sine is less than ·0001.

37. Repeat Expts. **35**, **36** for the graphs of $y = \tan x$, and $y = \sec x$.

From the above experiments it is seen that, provided the points are taken close enough together, the graphs of trigonometrical ratios can be considered to coincide with the series of short straight lines joining consecutive points; and that the closer these points are taken together, the less is the error introduced by this assumption. This law will be found to be, with certain restrictions, a general law for all graphs.

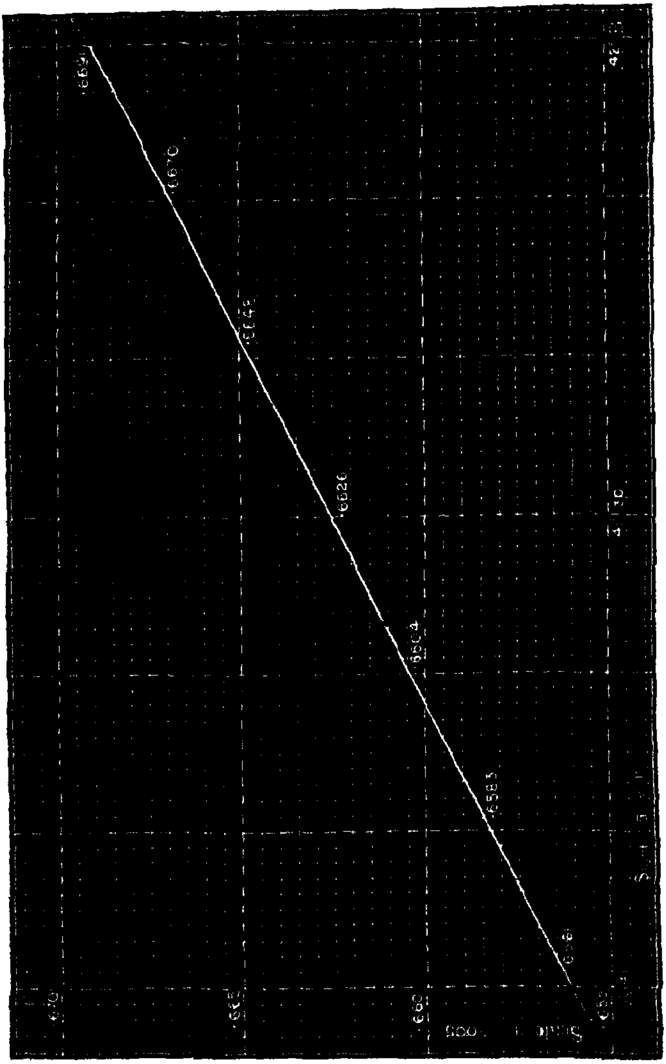


FIG. 38.

38. Draw two straight lines Ox, Oy at right angles to one another, and APQR a straight line inclined to them.

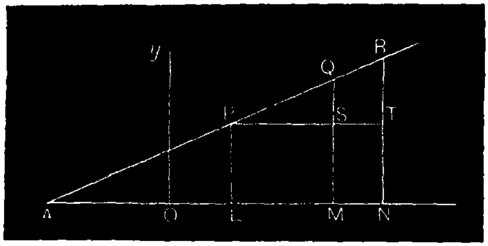


Fig 39

Draw PL, QM, RN perpendicular to Ox, and PST parallel to Ox, cutting QM, RN in S, T respectively. Then the angle RAN is the **angle of slope** of the line APQR: call this angle θ.

Then $\angle \mathrm{RPT} = \theta$.

$$\therefore \tan\theta = \frac{\mathrm{SQ}}{\mathrm{PS}} = \frac{\mathrm{MQ}-\mathrm{MS}}{\mathrm{LM}} = \frac{\mathrm{MQ}-\mathrm{LP}}{\mathrm{OM}-\mathrm{OL}}.$$

$$\text{Similarly } \tan\theta = \frac{\mathrm{NR}-\mathrm{LP}}{\mathrm{ON}-\mathrm{OL}}.$$

$$\therefore \frac{\mathrm{MQ}-\mathrm{LP}}{\mathrm{OM}-\mathrm{OL}} = \frac{\mathrm{NR}-\mathrm{LP}}{\mathrm{ON}-\mathrm{OL}}.$$

The results obtained in Expt. 38 are most important; they may be expressed in words as follows:

I. The ratio of the difference of the ordinates (y's) to the difference of the abscissae (x's) of two points is equal to the tangent of the angle of slope of the straight line joining the two points.

II. "The rate of increase" of the ordinate along a straight line is constant.

§ 11. Tabular Interpolation.

39. Draw two straight lines Ox, Oy at right angles to one another. Take any two points P, R and draw PL, RN per-

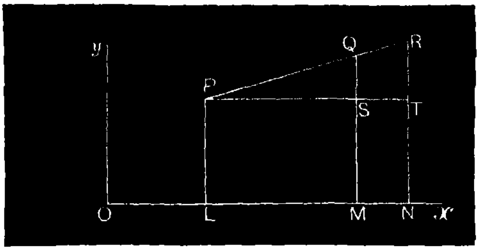


FIG 40

pendicular to Ox and PT perpendicular to RN Let OL, ON, LP, NR be x_1, x_2, y_1, y_2 respectively.

Take a point M on Ox, such that OM$=x$; draw MQ perpendicular to Ox, meeting PT in S and PR in Q. Let MQ$=y$

Then, by Expt. 38,

$$\frac{y_2-y_1}{x_2-x_1}=\tan \text{RPT}=m \text{ say},$$

$$\frac{y-y_1}{x-x_1}=\tan \text{RPT}=m$$

$$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}=m,$$

$$\therefore\ y=y_1+m(x-x_1)$$

This may be written

$$y=y_1+\frac{x-x_1}{x_2-x_1}(y_2-y_1) \qquad \ldots\ \ldots. .(1)$$

Similarly if x_1, y_1, x_2, y_2, and y are given,

$$x=x_1+m'(y-y_1) \text{ where } m'=\frac{x_2-x_1}{y_2-y_1},$$

$$\text{or}\ \ x=x_1+\frac{y-y_1}{y_2-y_1}(x_2-x_1) \quad \ldots\ldots \qquad \ldots.(2).$$

It follows that, if the values of x and y for two points on a straight line are known and the value of either x or y for any other point on the straight line is given, then the other value for the point can be found by either one or other of the formulae of Expt. **39**.

Also, since it was deduced from Expts. **35, 36, 37** that any graph could be considered to be composed of a large number of very short straight lines joining near points on it, these formulae can be used to obtain the values of the x or y of any point on the graph for which the values are not tabulated, if one of them is given. This process is however only used, as a rule, for points *intermediate* between the given points.

These deductions may be enunciated as a law, generally called the "**Rule of proportional differences**" or the "Principle of proportional parts."

RULE. In any graph, the differences between the ordinates of any three points on it are proportional to the corresponding differences between the abscissae, provided these differences are small compared respectively with the ordinates or abscissae themselves.

With the meaning, given above, assigned to the words "provided the points are close enough together," the Rule may be applied to **numerical or tabular interpolation** with a set of tables without drawing the graph.

40. Find $\sin 35^\circ 28' 30''$.

It is found from the tables that

$$\sin 35^\circ 20' = 0{\cdot}57833,$$
$$\sin 35^\circ 30' = 0{\cdot}58070;$$
$$\therefore \sin 35^\circ 30' - \sin 35^\circ 20' = 0{\cdot}00237.$$

But, by the "Rule of Proportional Differences,"

$$\frac{\sin 35^\circ 28' 30'' - \sin 35^\circ 20'}{\sin 35^\circ 30' - \sin 35^\circ 20'} = \frac{35^\circ 28' 30'' - 35^\circ 20'}{35^\circ 30' - 35^\circ 20'}$$

$$= \frac{8' 30''}{10'}$$

$$= \tfrac{17}{20};$$

$$\therefore \sin 35^\circ 28' 30'' - \sin 35^\circ 20' = \tfrac{17}{20} (\sin 35^\circ 30' - \sin 35^\circ 20')$$

$$= \tfrac{17}{20} \times \cdot 00237$$

$$= \cdot 00201;$$

$$\therefore \sin 35^\circ 28' 30'' = \sin 35^\circ 20' + \cdot 00201$$

$$= 0 \cdot 57833 + \cdot 00201$$

$$= \underline{\underline{0 \cdot 58034.}}$$

The *form of writing out* the different parts of the example worked above is important. It should be as short and concise as possible. The student should compare the following solution, step by step, with the above, noticing especially how the decimal points are omitted.

[*Model Solution.*]

$$\left.\begin{array}{ll} \sin 35^\circ 30' & = 0 \cdot 58070 \\ \sin 35^\circ 20' & = 0 \cdot 57833 \end{array}\right\} \text{diff. for } 10' = +237;$$

$$\therefore \text{diff. for } \quad 8' 30'' = \underline{\quad +201};$$

$$\therefore \sin 35^\circ 28' 30'' = \underline{\underline{0 \cdot 58034.}}$$

Note. The working out of the proportional difference for 8′ 30″ may be done by proportion as in the full-length solution above or by practice thus :—

$$\begin{array}{llr} & \text{diff. for } 10' & = 237 \\ \hline \therefore & \text{diff. for } \;\, 8' & = 189|6 \\ & \text{diff. for } \quad 30'' & = 11|85 \\ \hline \therefore & \quad\text{for } \;\, 8' 30'' & = 201|45 \end{array}$$

41. Find $\cos^{-1} 0{\cdot}32761 = x$ say.

It is found from the tables that

$$\cos 70^\circ 50' = 0{\cdot}32832,$$

$$\cos 71^\circ 0' = 0{\cdot}32557,$$

$$\text{and } \cos x^\circ = 0{\cdot}32761;$$

$$\therefore \frac{\cos x - \cos 70^\circ 50'}{\cos 71^\circ 0' - \cos 70^\circ 50'} = \frac{-{\cdot}00071}{-{\cdot}00275} = \frac{71}{275}.$$

But, by the "Rule of Proportional Differences,"

$$\frac{\cos x - \cos 70^\circ 50'}{\cos 71^\circ 0' - \cos 70^\circ 50'} = \frac{x - 70^\circ 50'}{10'};$$

$$\therefore \frac{x - 70^\circ 50'}{10'} = \frac{71}{275};$$

$$\therefore x = 70^\circ 50' + \tfrac{71}{275} \times 10'$$

$$= \underline{70^\circ 52' 35''}.$$

[*Model Solution.*]

$$\left.\begin{array}{l} \cos x \qquad\; = 0{\cdot}32761 \\ \cos 70^\circ 50' = 0{\cdot}32832 \end{array}\right\} \text{diff.} = -71,$$

$$\left.\begin{array}{l} \cos 70^\circ 50' = 0{\cdot}32832 \\ \cos 71^\circ 0' \;\; = 0{\cdot}32557 \end{array}\right\} \text{diff. for } 10' = -275;$$

$$\therefore x = 70^\circ 50' + \tfrac{71}{275} \times 10'$$

$$= \underline{70^\circ 52' 35''}.$$

42. Find $x = \tan^{-1} 2$.

[*Model Solution.*]

$$\left.\begin{array}{l} \tan x \qquad\; = 2{\cdot}00000 \\ \tan 63^\circ 20' = 1{\cdot}99117 \end{array}\right\} \text{diff.} = +883,$$

$$\left.\begin{array}{l} \tan 63^\circ 20' = 1{\cdot}99117 \\ \tan 63^\circ 30' = 2{\cdot}00569 \end{array}\right\} \text{diff. for } 10' = +1452;$$

$$\therefore x = 63^\circ 20' + \tfrac{883}{1452} \times 10'$$

$$= \underline{63^\circ 26' 5''}.$$

43. Find $\cot 63^\circ 26' 5''$.

[*Model Solution*]

$$\left.\begin{array}{l}\cot 63^\circ 30' = 0{\cdot}49858 \\ \cot 63^\circ 20' = 0{\cdot}50222\end{array}\right\} \text{diff for } 10' = -364;$$

$$\therefore \text{diff. for } 6' 5'' = \underline{-222};$$

$$\therefore \cot 63^\circ 26' 5'' = \underline{0{\cdot}50000}$$

EXERCISES

1 Find the values of the

(*a*) sine and tangent of $11^\circ 2' 37''$, $39^\circ 21' 26''$,

(*b*) cosine and cosecant of $55^\circ 27' 1''$, $82^\circ 7' 28''$,

(*c*) secant of $0^\circ 2' 37''$, $19^\circ 21' 37''$;

(*d*) cotangent of $61^\circ 20' 20''$, $79^\circ 13' 4''$

2 Find the values of

(*a*) $\sin^{-1}\frac{1}{2}$, $\sin^{-1}\frac{\sqrt{3}}{2}$, $\sin^{-1}\frac{\sqrt{3}-1}{2\sqrt{2}}$, $\sin^{-1}\frac{1}{\sqrt{2}}$, $\sin^{-1}\frac{\sqrt{5}-1}{4}$;

(*b*) $\sin^{-1}\frac{1}{4}$, $\sin^{-1}\frac{1}{3}$, $\cos^{-1}\frac{1}{5}$, $\cos^{-1}\frac{\sqrt{2}}{\sqrt{3}}$,

(*c*) $\tan^{-1}3$, $\cot^{-1}\sqrt{3}$, $\sec^{-1}2$, $\operatorname{cosec}^{-1}3$

3. Shew from the tables that

(1) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = 45^\circ$;

(2) $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99} = 45^\circ$;

(3) $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = 45^\circ$;

(4) $\cos^{-1}\frac{4}{5} - \sin^{-1}\frac{1}{\sqrt{10}} + \tan^{-1}\frac{1}{2} = 45^\circ$;

(5) $\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{3\sqrt{11}} + \sin^{-1}\frac{3}{\sqrt{11}} = 90^\circ$.

§ 12. Trigonometrical Ratios for an Obtuse Angle.

44. Draw a triangle ABC in which ∠s A and B are acute.

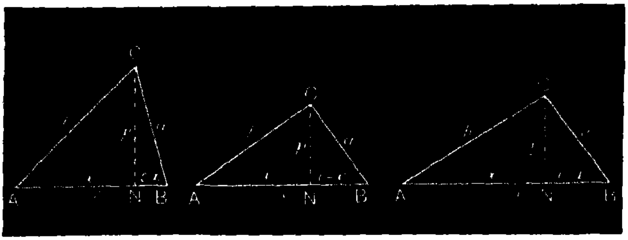


FIG. 41.

From C draw CN perpendicular to AB. Let the sides opposite the angles A, B, C be denoted by a, b, c respectively, CN by p, and area of $\triangle$ ABC by Δ.

Then $\Delta = \frac{1}{2}pc$

Hence shew that

$$\Delta = \tfrac{1}{2}\, bc \sin A = \tfrac{1}{2}\, ca \sin B.$$

The formulae obtained in Expt. **44** may be expressed in words thus:

The area of a triangle is measured by half the product of the lengths of two sides and the sine of the included angle:

so long as the included angle is *acute.*

45. If ABC is a triangle in which C is an obtuse angle shew that its area is given by $\frac{1}{2}ab \cdot \sin(180° - A)$.

46. In Fig. 42 let AN be denoted by x, then $NB = c - x$. Shew that

$$b^2 - x^2 = a^2 - (c - x)^2,$$

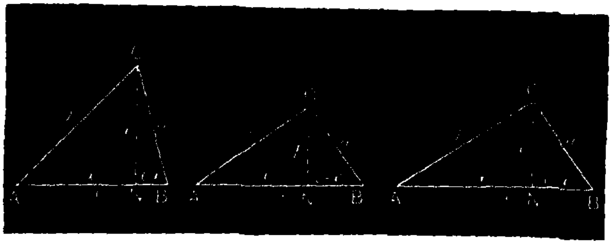


FIG. 42.

and obtain the formulae

$$b^2 + c^2 - a^2 = 2bc \cos A.$$
$$c^2 + a^2 - b^2 = 2ca \cos B.$$

The formulae obtained in Expt. **46** may be expressed in words thus:

The excess of the sum of the squares of the lengths of two sides of a triangle over the square of the length of the third side is equal to twice the product of the lengths of the two sides multiplied by the cosine of the included angle:

so long as the included angle is *acute*.

47. If ABC is a triangle in which C is an obtuse angle, shew that

$$c^2 - a^2 - b^2 = 2ab \cos (180^\circ - C).$$

It is convenient to *assume that the formulae obtained by Expts.* **44, 46** *are true whether the included angle is acute, a right angle or obtuse.*

In order to do this the definitions of the trigonometrical ratios must be extended, so that, if A is an obtuse angle,

$$\sin A = \sin (180° - A), \qquad \text{[Why ?]}$$

$$\cos A = -\cos (180° - A), \qquad \text{[Why ?]}$$

and

$$\sin 90° = 1, \quad \cos 90° = 0. \qquad \text{[Why ?]}$$

For the present these relations will be taken as definitions, the general definitions for angles of any magnitude being given later on.

EXERCISES.

1. Deduce from Expt. **44** that **the sides of any triangle are proportional to the sines of the opposite angles**, i.e. shew that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{2\Delta}{abc}.$$

2. Prove, geometrically or otherwise, that *in any triangle*

$$a = b \cos C + c \cos B,$$

$$b = c \cos A + a \cos C,$$

$$c = a \cos B + b \cos A.$$

3. Deduce from the formulae of Ex. 2 that *in any triangle*

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

4. Shew that

$$\sin A = \frac{\sqrt{2(b^2c^2 + c^2a^2 + a^2b^2) - a^4 - b^4 - c^4}}{2bc},$$

and, hence,

$$\Delta = \tfrac{1}{4}\sqrt{2(b^2c^2 + c^2a^2 + a^2b^2) - a^4 - b^4 - c^4}.$$

5. If A, B, C are the angles of a triangle, shew that

$$\sin(A+B) = \sin C = \sin A\cos B + \cos A\sin B,$$
$$\cos(A+B) = -\cos C = \cos A\cos B - \sin A\sin B,$$

by substituting the values obtained in Exs. 3, 4.

6. Verify the formula of Ex. 1 for a triangle whose sides are $a=17$, $b=44$, $c=39$; and find its area.

[From Expt. 44, $\cos A = \dfrac{44^2+39^2-17^2}{2 . 44 . 39} = \dfrac{12}{13}$;

$$\therefore \sin A = \sqrt{1-(\tfrac{12}{13})^2} = \tfrac{5}{13}.$$

Similarly, $\sin B = \tfrac{220}{221}$,

$\sin C = \tfrac{15}{17}$;

$$\therefore \sin A : \sin B : \sin C = \tfrac{5}{13} : \tfrac{220}{221} : \tfrac{15}{17}$$
$$= 17 : 44 : 39$$
$$= a : b : c.$$

Also $\Delta = \dfrac{1}{2} ab \sin C$

$$= \frac{1}{\cancel{2}} \cdot \cancel{17} \cdot \overset{22}{\cancel{44}} \cdot \frac{15}{\cancel{17}}$$
$$= 330\ ]$$

7. Find the areas of the triangles whose sides are as follows, stating in each case which, if any, is the obtuse angle:

(1) $a=25$, $b=26$, $c=17$;

(2) $a=50$, $b=80$, $c=78$;

(3) $a=15$, $b=26$, $c=37$.

8. A triangular field has its sides respectively 242, 1212 and 1450 yards long: shew that its area is 6 acres.

9. Assuming the "Triangle of Forces" for three forces in equilibrium, shew that

"Each force is proportional to the sine of the angle between the other two" [Lami's Theorem].

§ 13. "Solution of Triangles."

Hitherto, when solving problems on "Heights and Distances" by two observations, the triangle drawn to represent the given conditions of the problem has been solved by drawing a perpendicular breaking up the triangle into the sum or difference of two *right-angled triangles*, and the parts of these have been calculated by means of the tables on pp. 36—41, and hence the parts of the original triangle have been deduced.

The formulae obtained in § 12 can, however, be used to solve the triangle, if it is carefully remembered that in the case of an obtuse angle X,

$$\sin X = \sin(180° - X),$$
$$\cos X = -\cos(180° - X).$$

Formula I. (*"The rule of sines."*)

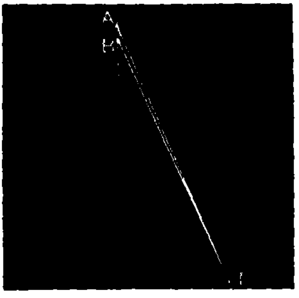
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

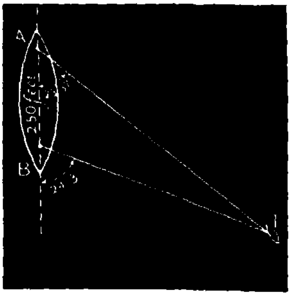
This formula can be used when any *one* fraction is completely known—say, A and a are given—and one other part, either an angle, as B, or a side as b, is also known.

Note. It will be found, as a rule, that, if an attempt is made to draw a diagram to scale, a figure is obtained (Fig. 43 *a*), which is useless for *practical purposes*, and not clear enough for illustrating a theoretical solution of the problem. Hence generally for *theoretical solution* of problems, a distorted diagram (Fig. 43 *b*) is drawn.

A. Two angles and one side.

48. A man-of-war has two big guns, one fore and the other aft. A torpedo-boat is sighted simultaneously by the captains of the guns, the telescopes making angles of 23° 37′ and 24°3′ with the line from stem to stern. Given that the distance between the guns is 250 ft., find the correct range for each gun.





(*a*) Fig. 43. (*b*)

In the triangle ATB, we know two angles (and therefore the third) and one side: and we have also that

$$\sin \mathrm{TBA} = \sin(180° - 24°3') = \sin 24°3',$$

and $$\sin \mathrm{ATB} = \sin(24°3' - 23°37') = \sin 26'.$$

Hence, by Formula I,

$$\frac{\sin 26'}{250} = \frac{\sin 24°3'}{\mathrm{AT}} = \frac{\sin 23°37'}{\mathrm{BT}},$$

$$\therefore\ \mathrm{AT} = \frac{250 \times \sin 24°3'}{3 \times \sin 26'} \text{ yds.} = 4492 \text{ yds.},$$

$$\mathrm{BT} = \frac{250 \times \sin 23°37'}{3 \times \sin 26'} \text{ yds.} = 4416 \text{ yds.}$$

Note. The average distance from any part of the ship is thus 4450 yds. The above solution is the fundamental idea of Lord Kelvin's patent range-finder; the telescopes are connected with a Wheatstone bridge arrangement and automatically register the average range on a dial in the conning-tower.

B. Two sides and the angle opposite one of them.

49. A cyclist is touring in a district for which he has a very imperfect map. The chief towns of the district A, B and C are connected by straight roads, and B and C bear E.N.E. and due E. from A. He rides from A to B and on to C, arriving at 2 p.m.; from the cyclometer he finds it is 25 miles from A to B, and 12 miles from B to C. At what time can he get back to A, riding at an average speed of 10 miles an hour?

In this problem we have

$$A = 22°30', \quad a = 12 \text{ miles}, \quad c = 25 \text{ miles}.$$

Hence, by Formula I,

$$\frac{\sin 22°30'}{12} = \frac{\sin C}{25} = \frac{\sin B}{AC},$$

$$\therefore \sin C = \tfrac{25}{12} \times \sin 22°30' = \sin 52°52'.$$

Since $\sin X = \sin(180 - X)$, it cannot be determined from the above calculation whether $C = 52°52'$ or $127°8'$.

(i) If $C = 52°52'$, $B = 180° - 52°52' - 22°30' = 104°38'$,

$$\therefore AC = \frac{12 \times \sin 104°38'}{\sin 22°30'} \text{ miles}$$

$$= \frac{12 \times \sin 75°22'}{\sin 22°30'} \text{ miles}$$

$$= 30{\cdot}34 \text{ miles},$$

and the cyclist will get back to A about 5.2 p.m.

(ii) If $C = 127°8'$, $B = 180° - 127°8' - 22°30' = 30°22'$;

$$\therefore AC = \frac{12 \times \sin 30°22'}{\sin 22°30'}$$

$$= 15{\cdot}85 \text{ miles},$$

and the cyclist will get back to A at 3.35 p.m.

Thus there are two positions of C, such that the conditions of the problem are satisfied. This can well be shewn by means of a diagram drawn to scale.

Lay down the line **AC** due east and west, and the line **AB** towards E.N.E., making **AB** 25 miles to some convenient scale. Then a circle, with a radius equal to 12 miles on this scale, drawn with centre **B**, will cut **AC** in the two points which represent the two possible positions of **C**.

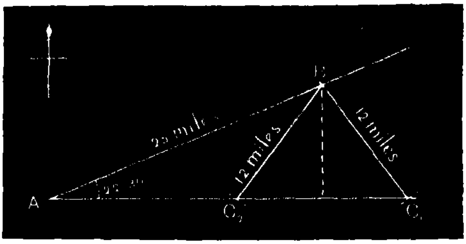


FIG. 44.

From Fig. 44 it is evident that if a is of any length *greater than the perpendicular from* **B** *to* **AC**, *i.e. greater* than $c \sin A$, the circle will intersect **AC** in two points and the solution of the problem will be **ambiguous.**

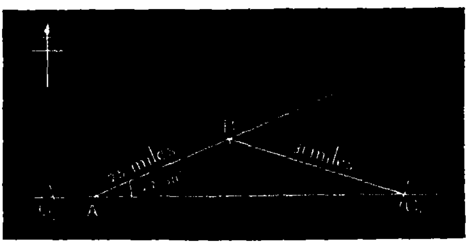


FIG. 45.

If, however, $a > c$, as in Fig. 45, one of these points (C_2) will be on the left-hand side of **A**, and the triangle ABC_2 will not contain the angle 22° 30′, the angle at **A** being

$$180° - 22° 30' = 157° 30'.$$

In the trigonometrical solution this case is determined by the fact that of the two values for **C** that are found from the value of sin **C**, the sum of one of them and the given angle **A** is not less than 180°, and thus no triangle is possible having these angles as two of its angles.

50. In Expt. **49** suppose the cyclist to have found that BC was 31 miles instead of 12 miles. Find his time of return to A as before.

We have

$$\sin C = \tfrac{25}{31} \times \sin 22^\circ 30' = \sin 17^\circ 59';$$

$$\therefore \text{ C is either } 17^\circ 59' \text{ or } 162^\circ 1'.$$

The sum of A and C is either 40° 29′ or 184° 31′, *i.e.* there is only one value of B: and we have

$$\sin B = \sin (180^\circ - 40^\circ 29') = \sin 40^\circ 29'.$$

$$\therefore \quad AC = \frac{31 \times \sin 40^\circ 29'}{\sin 22^\circ 30'}$$

$$= 52{\cdot}59 \text{ miles,}$$

and the cyclist would get back to A at 7.16 p.m.

This case is of little practical importance, but the first case is extremely important.

In survey work a "base line," as long as can conveniently be obtained is measured with extreme accuracy. The base line for English Survey on Salisbury Plain is, for instance, over 7 miles long, and the error is not more than 3 in. From the ends of this base line the bearings of another point are accurately measured with a theodolite and the triangle is solved: this gives two more lines which may be used as base-lines to find the positions of other points, and thus step by step the whole country can be covered with a series of accurately surveyed triangles, and maps can be drawn on a reduced scale.

Formula II. (*"The cosine rule."*)

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

or

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \ldots\ldots (2).$$

This formula in its first form can be used to find the angles of a triangle when the lengths of the three sides are given; in the second form, when the lengths of two sides and the magnitude of the included angle are given the third side can be found, and then the other angles can be calculated by "the rule of sines."

C. Two sides and the contained angle.

51. Two forces of 7 and 19 lbs. wt. act on a body, the angle between their directions being 43°. Find the magnitude of their resultant.

From the "vector law" it is known that, if OA, OB are drawn so that OA = 7, OB = 19 *on some convenient scale* and $\angle$ AOB = 43°, the resultant is represented by OC the diagonal of the parallelogram OACB.

Hence, in the triangle OAC,

$$OA = 19, \quad AC = 7, \quad \angle OAC = 180° - 43°.$$

Therefore, by Formula II,

$$\begin{aligned} OC^2 &= OA^2 + AC^2 - 2OA \,.\, AC \cos(180° - 43°) \\ &= OA^2 + AC^2 + 2OA \,.\, AC \cos 43° \\ &= 19^2 + 7^2 + 2 \,.\, 19 \,.\, 7 \times (\cdot 73135) \\ &= 361 + 49 + 194\cdot 5391 \\ &= 600\cdot 5391; \end{aligned}$$

$\therefore$ OC = 24·5.

$\therefore$ Resultant is a force of $24\frac{1}{2}$ lbs. wt.

D. Three sides.

52. The lengths of the sides of a triangular tract of land are 102, 195, 279 chains respectively. Calculate the magnitude of the greatest angle, and thence, by means of the formula

$$\Delta = \tfrac{1}{2}ab \sin C,$$

find the area of the land.

Let $a=102$, $b=195$, $c=279$ chains respectively.

Then, by Formula II,

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$= \frac{102^2+195^2-279^2}{2 \,.\, 102 \,.\, 195}$$

$$= -\cdot 73936.$$

From the tables $\cos 42° 19' 23'' = \cdot 73936$,

$$\therefore \; C = 180° - 42° 19' 23'' = 137° 40' 37''.$$

Again, $\quad \sin C = \sin 42° 19' 23''$

$$= \cdot 67331 \text{ from the tables;}$$

$$\therefore \; \text{area of land} = \frac{102 \times 195 \times \cdot 67331}{2} \text{ sq. chains}$$

$$= 669 \cdot 6 \text{ acres.}$$

Exercises.

1. Draw an equilateral triangle ABC; draw AD perpendicular to BC: shew that the sides of the triangle ABD are in the ratio $1 : 2 : \sqrt{3}$. Hence prove

(1) $\sin 60° = \cos 30° = \dfrac{\sqrt{3}}{2}$, $\qquad$ (3) $\tan 60° = \sqrt{3}$,

(2) $\cos 60° = \sin 30° = \frac{1}{2}$. $\qquad$ (4) $\tan 30° = \frac{1}{3}\sqrt{3}$.

2. The sides of a triangle are $2\sqrt{2}$, $2\sqrt{3}$, $\sqrt{6}-\sqrt{2}$: shew that its angles are 120°, 45° and 15°. Hence find sin 15°, cos 15°, tan 15°, sin 75°, cos 75°, tan 75° in the form of surds.

3. In a triangle ABC, AB = AC = 2, BC = $\sqrt{5}-1$: along BA mark off BD = $3-\sqrt{5}$, and apply the "sine rule" to shew that the triangles ABC, CBD are equiangular: hence shew that $\angle B = \angle C = 72°$ and $A = 36°$ and, using the "cosine rule," shew that

$$(1) \quad \cos 72° = \sin 18° = \frac{\sqrt{5}-1}{4},$$

$$(2) \quad \cos 36° = \sin 54° = \frac{\sqrt{5}+1}{4}.$$

Exercises 4—10 *should be worked* without *the use of the tables on pp.* 36—39.

4. The sides of a triangle are as $2 : \sqrt{6} : 1+\sqrt{3}$, find the angles.

5. The sides of a triangle are as 7 : 8 : 13, find the greatest angle.

6. Given $C = 18°$, $a = \sqrt{5}+1$, $c = \sqrt{5}-1$, solve the triangle.

7. Given $A = 75°$, $B = 30°$, $b = 2\sqrt{2}$, solve the triangle.

8. Given $B = 30°$, $c = 150$, $b = 50\sqrt{3}$, shew that, of the two triangles which satisfy these data, one will be isosceles and the other right-angled: find the length of the equal sides of the isosceles triangle.

9. Given $B = 15°$, $b = \sqrt{3}-1$, $c = \sqrt{3}+1$, solve the triangle.

10. The cosines of two of the angles of a triangle are respectively $\frac{1}{2}$ and $\frac{3}{5}$; find the ratio of the sides.

The following exercises are intended to be worked with the use of the tables on pp. 36—39.

11. The sides of a triangle are 17, 20, 27. Find all the angles.

12. Find the area of the triangle in Ex. 11, by means of the formula of Ex. 1, p. 62.

13. The angles of a triangle are in the ratio of 36 : 49 : 64: the smallest side is 6 inches long, find the other two.

14. One side of a triangle is 153 ft. long, a second side is 69 ft. long, and the angle opposite this side is $22^\circ 17' 20''$: find the difference between the two values which are obtained for the third side in the two triangles that can be drawn to satisfy these data.

15. Two forces P, Q are balanced by a third force R: $P=27$ lbs. wt., $Q=33$ lbs. wt., $R=56$ lbs. wt. Find the angle between P and Q.

16. Two posts A and B, 500 yds. apart, are set up on the sea-shore near a fort. A man-of-war (M) appears out at sea. The angles MAB, MBA are observed to be $76^\circ 22' 30''$ and $102^\circ 25' 45''$. Find the distance of the ship from each post.

17. Two sides of a triangle are 11 and 12 and the included angle is $45^\circ 48' 56''$. Find the length of the median bisecting the third side.

18. I stand on the bank of a river directly opposite a tree on the other bank, the elevation of the top of which is $25^\circ 30'$. I walk off at right angles to the line of sight along the bank of the river for 20 yds. when the elevation of the top of the tree becomes $24^\circ 45'$. Find the width of the river.

19. If m is the length of the median bisecting the side BC of a triangle ABC, shew that

$$4m^2 = 2b^2 + 2c^2 - a^2.$$

20. At each end of a horizontal line, 500 yds. long, the elevations of the top of a distant hill are 35° and 37°, whilst the elevation at the middle point of the line is $36^\circ 15'$. Find the height of the hill.

§ 14. Tables of "Indices."

All calculations mainly involving multiplications and divisions can be much simplified by the use of a table of "indices" of the powers of some chosen number.

DEF. A power of a number is the continued product of two or more factors each equal to the given number.

The number of factors in the continued product is indicated by a small figure placed above and to the right of the number—in "the postage-stamp corner."

This small figure is variously called an **exponent**, an **index**, or a **logarithm**.

53. Plot a series of points to shew the values of $y=(1{\cdot}2)^x$ for $x=1, 2, 3, \ldots 10$.

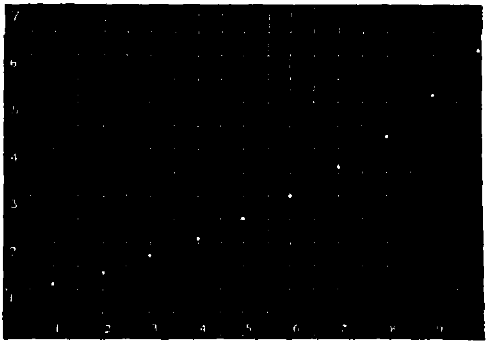


FIG. 46.

The "graph" consists of a number of isolated points corresponding to the *integral* values assigned to x; such an expression as $(1{\cdot}2)^{\frac{1}{2}}$ having no meaning according to the definition given above.

54. Join the isolated points obtained for the graph of $y=(1\cdot2)^x$ for integral values of x with a freehand curve.

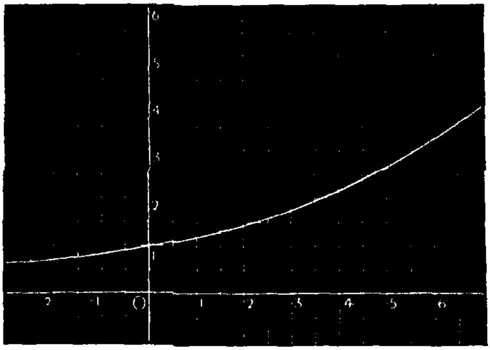


FIG. 47.

The intermediate points on this curve give us fractional, and points on its continuation to the left negative, values of x, although the above definition of an index gives us no meaning for the values of y for these points as 'powers' of 1 2. However, as the curve is *continuous*, it is most probably the locus of a point moving according to some Geometrical or Algebraical law, and, if this law can be found, it may be possible to assign meanings to the values of y when x is not an integer.

It follows at once from the given definition of a power that if a^m and a^n are two powers of a, then

$$a^m \times a^n = (a \times a \times a \times \dots m \text{ factors}) \times (a \times a \times a \times \dots n \text{ factors})$$
$$= a \times a \times a \times \dots \overline{m+n} \text{ factors}$$
$$= a^{m+n}.$$

Or graphically,

"If (x_1, y_1), (x_2, y_2) are two points on the graph of powers of a, where x_1, x_2 are integers, then $(x_1+x_2, y_1 \times y_2)$ is another point on the graph."

55. Take points on the graph of $y=(1\cdot2)^x$, (drawn as in Expt. **54**, but on as large a scale as possible), for which x is not a whole number; verify that, for any values of x_1, x_2 *positive or negative, integral or fractional*, such as those in the table below, if (x_1, y_1), (x_2, y_2), (x_3, y_3) are points on the curve such that $x_3=x_1+x_2$, then $y_3=y_1y_2$.

$x_1=$	1·2	·3	2·1	·9	·4	3·6	−4·0	−·9	−2·3	−7·2	−10
$x_2=$	1·3	1·7	2·6	7·3	·9	5·2	−2·8	−3·2	9·7	0	10
$x_3=$	2·5	2·0	4·7	8·2	1·3	8·8	−6·8	−4·1	7·4	−7·2	0
$y_1=$											
$y_2=$											
$y_3=$											
$y_1y_2=$											

56. Assume that this "Index Law," i.e. $a^m \times a^n = a^{m+n}$ is true not only for integral values of m and n, but *for all values, positive and negative*, as in the case of the graphical equivalent in Expt. **55**; find a meaning for $a^{\frac{1}{n}}$.

$$\text{Let} \quad a^{\frac{1}{n}} = x.$$

$$\text{Then } x^n = x \times x \times x \times \dots n \text{ factors,}$$

$$= a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \dots n \text{ factors,}$$

$$= a^{\frac{1}{n}+\frac{1}{n}+\dots n \text{ terms}},$$

$$= a^{\frac{n}{n}} = a,$$

$$\therefore\ x = \sqrt[n]{a},$$

i.e. $a^{\frac{1}{n}}$ **is a symbol standing for** $\sqrt[n]{a}$.

57. Find a meaning for the symbol $a^{\frac{p}{q}}$, shewing that it can be written in either of the forms

$$\sqrt[q]{a^p} \text{ or } (\sqrt[q]{a})^p.$$

58. Find a meaning for the symbol a^0.

[We have $a^n \times a^0 = a^{n+0} = a^n$, hence a^0 stands for 1.]

59. Find a meaning for the symbol a^{-n}, where n is integral or fractional.

[We have $a^n \times a^{-n} = a^{n-n} = a^0 = 1$, hence a^{-n} stands for $1 \div a^n$.]

60. Plot the graph of $y = 10^x$, as follows :—

(i) Find $10^{\frac{1}{2}}$ $(\sqrt{10})$, $10^{\frac{1}{4}}$ $(\sqrt{10^{\frac{1}{2}}})$, $10^{\frac{1}{8}}$ $(\sqrt{10^{\frac{1}{4}}})$, $10^{\frac{1}{16}}$ $\sqrt{10^{\frac{1}{8}}}$;

(ii) Obtain $10^{\frac{3}{16}}$, $10^{\frac{5}{16}}$, $10^{\frac{3}{8}}$, $10^{\frac{7}{16}}$, and higher powers by decimal approximation.

(iii) Plot on a large sheet of squared paper, taking $1'' = 1$ along the y-axis, and $10'' = 1$ along the x-axis.

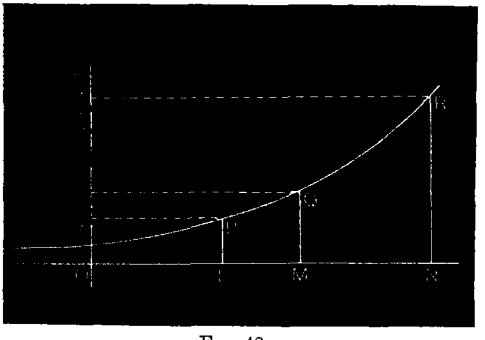


FIG. 48.

61. Use the graph of $y = 10^x$ to multiply 3·7 by 2·4.

Find points P and Q on the curve of which the ordinates PL and QM are 2·4 and 3·7. Along Ox mark off MN towards the right $=$ OL, so that ON $=$ OL $+$ OM ; draw the ordinate RN and measure it.

Verify that RN $= 3{\cdot}7 \times 2{\cdot}4$.

62. Use the graph of $y=10^x$ to obtain the quotients

$$\frac{3\cdot7}{2\cdot4} \text{ and } \frac{2\cdot4}{3\cdot7}.$$

[For $\frac{3\cdot7}{2\cdot4}$, along OX mark off MN, towards the *left*, equal to OL so that ON = OM − OL.]

63. Obtain from the graph of $y=10^x$ the values of $10^{\cdot1}, 10^{\cdot2}, 10^{\cdot3}, 10^{\cdot4} \ldots 10^{\cdot9}$; and find, by the method of interpolation explained in § 11, the values of

$$10^{\cdot07},\ 10^{\cdot14},\ 10^{\cdot21},\ 10^{\cdot35},\ 10^{\cdot49},\ 10^{\cdot56},\ 10^{\cdot63},\ 10^{\cdot77},\ 10^{\cdot84},\ 10^{\cdot98};$$

plot the corresponding points in connection with the curve and, by noticing how near to the curve the points lie, observe for what values of x and to what extent the "Rule of proportional differences" is to be trusted with this graph or *with a corresponding set of tables.*

64. Draw as much as possible of a graph of $y=10^x$ on a large sheet of paper, for values of x between 2·2 and 2·4 taking $10''=1$ along the x axis and $1''=1$ along the y axis, and examine to what extent the "Rule of proportional differences" applies for these values of x.

1. Find from the graph of $y=10^x$ the values of

$$(a)\ \frac{5\cdot3\times2\cdot7}{8\cdot6}, \quad (b)\ \frac{\cdot76}{1\cdot03\times2\cdot42},$$

$$(c)\ (1\cdot33)^3, \quad (d)\ (2\cdot17)^2.$$

2. Extract the square roots of 3, 5, 6, 7; i.e., find values of $3^{\frac{1}{2}}, 5^{\frac{1}{2}}, 6^{\frac{1}{2}}, 7^{\frac{1}{2}}$.

3. Extract the cube roots of 2, 4, 9.

§ 15. Tables of Logarithms.

It appears from the results of Expts. **63, 64**, that *provided the differences are small compared with the values of x chosen*, the rule of Proportional differences is applicable to the graph of $y = 10^x$, or to the corresponding tables.

By methods of more advanced Trigonometry (*see Lock's Elementary Trigonometry* (1903), pp. 246—250), tables can be constructed, accurate to any required number of decimal places, giving the values of the index or logarithm x for consecutive values of y. A set of such tables calculated to five places of decimals is given on pp. 89, 90.

When using the tables, the student should always bear in mind that a logarithm is an index.

DEF. (a). **The logarithm of a number N is the index of that power of some number a which is equal to N; the number a is called the base of the system of logarithms,**

or making use of symbols,

DEF. (b). If $a^l = \mathsf{N}$, then l is the logarithm of N to the base a. The logarithm of N to the base a is usually written $\log_a \mathsf{N}$.

Thus, $$a^{\log_a \mathsf{N}} = \mathsf{N}.$$

Hence it is obvious that the laws of indices apply to logarithms; in fact, the laws should be called the "laws of logarithms," the name "index" being retained as a special name for a logarithm which happens to be an integer.

65. Given $\log_a \mathbf{m} \equiv \mathbf{x}$, $\log_a \mathbf{n} \equiv \mathbf{y}$; find $\log_a (\mathbf{m} \times \mathbf{n}) \equiv \mathbf{z}$

By Def. (*b*) on p. 78, we have

$$a^x \equiv m, \quad a^y \equiv n, \quad a^z \equiv m \times n,$$

$$\therefore \; a^z = a^x \times a^y = a^{x+y},$$

$$\therefore \; z = x + y,$$

or, more directly,

$$a^{\log_a (m \times n)} = m \times n = a^{\log_a m} \times a^{\log_a n} = a^{\log_a m + \log_a n}.$$

$$\therefore \; \mathbf{\log_a (m \times n) = \log_a m + \log_a n.}$$

1. Shew that

$$\log_a (p.q.r.s.t) = \log_a p + \log_a q + \log_a r + \log_a s + \log_a t.$$

2. Given $\log_{10} 2 = 0{\cdot}30103$, $\log_{10} 3 = 0{\cdot}47712$,

find $\log_{10} 4$, $\log_{10} 6$, $\log_{10} 8$, $\log_{10} 12$, $\log_{10} 16$, $\log_{10} 18$.

3. **Shew that $\log_a (\mathrm{m}^n) = (\log_a \mathrm{m}) \times \mathrm{n}$.**

4. Find $\log_{10} 27$, $\log_{10} 32$, $\log_{10} 64$, $\log_{10} 81$.

5. **Shew that $\log_a \sqrt[n]{\mathrm{m}} = (\log_a \mathrm{m}) \div \mathrm{n}$.**

6. Find $\log_{10} \sqrt{2}$, $\log_{10} \sqrt[3]{3}$, $\log_{10} \sqrt[3]{9}$, $\log_{10} \sqrt[2]{12}$.

7. **Shew that $\log_a \dfrac{\mathrm{m}}{\mathrm{n}} = \log_a \mathrm{m} - \log_a \mathrm{n}$.**

8. Shew that

$$\log_a \frac{p.q.r}{s.t} = \log_a p + \log_a q + \log_a r - \log_a s - \log_a t.$$

9. **From the given values of $\log_{10} 2$ and $\log_{10} 3$ above find the logarithms to the base 10 of 5, 15, $1\frac{1}{2}$, $7\frac{1}{2}$, 1·25, 3·6, 4·8.**

Shew that $\log_a a^n = n$:

d the values of $\log_{10} 1$, $\log_{10} 10$, $\log_{10} 100$, $\log_{10} 1000$.

§

1. Given $\log_{10} 2$ and $\log_{10} 3$ above, and that $\log_{10} 7 = \cdot 84510$ $\log_{10} 120$, $\log_{10} 126$: from these by interpolation find $\log_{10} 125$, vid compare it with the value obtained from $\log_{10} 5$: examine to what extent the Rule of Proportional Differences is applicable for differences as large as this.

12. (a) Take from the tables on pp. 89, 90, $\log_{10} 100$, $\log_{10} 101$ and interpolate for 100·76; (b) take $\log_{10} 100 \cdot 7$, $\log_{10} 100 \cdot 8$ and interpolate for 100·76: compare the results obtained in (a) and (b).

deci

x for co Find $\log_{10} 250$, $\log_{10} 252$ and, by interpolation, $\log_{10} 25088$: lated to 25088 by putting $25088 = 2^9 . 7^2 . 7$, and examine to what W e Rule of Proportional Differences is applicable here.

bear

From the answers to questions 11, 12, 13 above, it will be found that, when using five-figure logarithms, the Rule of Proportional Differences can be safely applied between 1·1000 and 9·9999, when the differences are less than ·01, for *two more significant* figures, but that between 1·00 and 1·10 the differences must be less than ·001 for safety. Hence, the tables on pp. 89, 90 are given for four significant figures, with one extra to be calculated by interpolation, for numbers between 1·0000 and 1·1000, and for three significant figures, with two extra to be calculated, for numbers between 1·1000 and 9·9999.

66. Find $\log_a (\mathrm{N} \times a^n)$, given $\log_a \mathrm{N}$.

We have
$$\log_a (\mathrm{N} \times a^n) = \log_a \mathrm{N} + \log_a a^n,$$
$$= \log_a \mathrm{N} + n \log_a a,$$
$$\therefore\ \boldsymbol{\log_a (\mathrm{N} \times a^n) = \log_a \mathrm{N} + n.}$$

Also shew that

$$\log_a (N \div a^n) = \log_a N - n.$$

Hence
$$\log_{10}(N \times 10) = \log_{10} N + 1,$$
$$\log_{10}(N \times 10^2) = \log_{10} N + 2,$$
$$\log_{10}(N \div 10) = \log_{10} N - 1,$$
$$\log_{10}(N \div 10^2) = \log_{10} N - 2;$$
and so on.

Thus it appears that any two decimal numbers, *expressed by the same significant figures in the same order*, have logarithms to base **10** which *differ only by whole numbers*; that is they have the decimal part of their logarithms—called the **mantissa**—identical. For multiplication by 10 only moves the decimal point in the number one place to the right, leaving the sequence of significant figures unaltered; whilst it adds on $\log_{10} 10$ $(= 1)$ to the logarithm, thus leaving the mantissa unaltered.

Any number can be expressed as the product of a number between 1 and 10 and a power of 10. This operation is called "reducing the number to **Standard Form**."

Thus
$$24696 = 2{\cdot}4696 \times 10^4,$$
$${\cdot}0024696 = 2{\cdot}4696 \times 10^{-3}.$$

When a number is reduced to Standard Form, not only is the real value of each significant figure much more easily recognized, but also its logarithm is more easily determined.

14. Given $\log_{10} 3425 = 3{\cdot}53466$, find

$$\log_{10} 34{\cdot}25, \ \log_{10} {\cdot}03425.$$

67. Given $\log_{10} 2 = \cdot 30103$, $\log_{10} 3 = \cdot 47712$, $\log_{10} 7 = \cdot 84510$, find $\log_{10} 24696$, $\log_{10} 2\cdot 4696$, $\log_{10} \cdot 0024696$.

[Since $24696 = 2^3 . 3^2 . 7^3$;

$$\therefore \log_{10} 24696 = 3 \log_{10} 2 + 2 \log_{10} 3 + 3 \log_{10} 7,$$

$$\begin{aligned} &= \ \ 0\cdot 90309 \\ &\quad +0\cdot 95424 \\ &\quad +2\cdot 53530 \\ &= \ \ 4\cdot 39263. \end{aligned}$$

Now $$24696 = 2\cdot 4696 \times 10^4,$$

$$\therefore \log 2\cdot 4696 = 0\cdot 39263.$$

Again, $$\cdot 0024696 = 2\cdot 4696 \times 10^{-3},$$

$$\begin{aligned} \therefore \log \cdot 0024696 &= \log 2\cdot 4696 - 3 \\ &= 0\cdot 39263 - 3.] \end{aligned}$$

It follows from Expt. **66** that, if the logarithm of 2·4696 is known, the logarithms of all numbers having this sequence of significant figures can be written down immediately. Hence, in constructing a system of tables to the base 10 only the mantissae for the sequences of figures need be given, the integral part of the logarithm—called the **characteristic**—being attached according to the following rule.

RULE. **The characteristic of the logarithm of a number is the index of the power of 10 which appears as a factor when the number is brought to standard form.**

15. Express the following numbers in standard form, and find the characteristics of their logarithms :—

3·672, 230·75, 0·023, 0·3, 0·00001.

16. Find, without actually calculating the number itself, the "place-value" of the first significant figure in

$$2^{32},\ 16^7,\ \sqrt{0\cdot 3},\ \sqrt[10]{1000}.$$

The use of tables of Logarithms necessitates keeping the mantissa positive, even for the logarithms of such numbers as $\cdot 0024696$. It is inconvenient to write the logarithm of this number as $0\cdot 39263 - 3$ or $-3 + 0\cdot 39263$; whilst it is incorrect to write it as $-3\cdot 39263$, which would stand for $-3 - 0\cdot 39263$. Hence the **minus sign is written over the characteristic** to show that it alone, and not the mantissa, is negative; thus,

$$\log_{10} \cdot 0024696 = \bar{3}\cdot 39263 \text{ (read "bar three, point, etc.")}.$$

This must be most carefully remembered, especially in calculating logarithms of square, cube and other *roots* of numbers.

68. Find $\log_{10}\sqrt{\cdot 0024696}$, $\log_{10}\sqrt[5]{\cdot 0024696}$.

[We have
$$\begin{aligned}\log_{10}\sqrt{\cdot 0024696} &= \tfrac{1}{2}\log_{10}\cdot 0024696\\ &= \tfrac{1}{2}(\bar{3}\cdot 39263)\\ &\left\{\begin{aligned}&= \tfrac{1}{2}(-3+\cdot 39263)\\ &= \tfrac{1}{2}(-4+1\cdot 39263)\\ &= -2+0\cdot 69632\end{aligned}\right\}\\ &= \bar{2}\cdot 69632.\end{aligned}$$

Again,
$$\begin{aligned}\log_{10}\sqrt[5]{\cdot 0024696} &= \tfrac{1}{5}\log_{10}\cdot 0024696\\ &= \tfrac{1}{5}(\bar{3}\cdot 39263)\\ &\left\{\begin{aligned}&= \tfrac{1}{5}(-3+0\cdot 39263)\\ &= \tfrac{1}{5}(-5+2\cdot 39263)\\ &= -1+0\cdot 47853\end{aligned}\right\}\\ &= \bar{1}\cdot 47853.\end{aligned}$$]

Note. The three lines of working enclosed by the brackets represent mental work. The negative characteristic is increased until it becomes a multiple of the divisor, and at the same time the positive mantissa is increased by the same integer; the division is then performed in two parts, and the quotients once more are associated.

Tables of the logarithms of the trigonometrical ratios can also be constructed ; and the rule of proportional differences, being true for the natural trigonometrical ratios (§ 11), and also for logarithms (§ 15), is true for the logarithms of the ratios.

Sines and cosines are always less than unity, as also are the tangents of all angles between 0° and 45°. The logarithms of these Ratios must therefore have *negative* characteristics.

To avoid the inconvenience of having to print these negative characteristics, the whole number 10 is added to each logarithm of the Trigonometrical Ratios, before it is set down in the Table. The numbers thus recorded are called the **tabular logarithms** of the sine, cosine, etc., of an angle.

Thus opposite 31° 15′ in the logarithmic tables of sines we find 9·71498,

$$\therefore \log \sin 31^\circ 15' = 9{\cdot}71498 - 10 = \bar{1}{\cdot}71498.$$

Tabular logarithms are indicated by the letter '*L*.' Thus $L\sin 31^\circ 15'$ stands for the tabular logarithm of $\sin 31^\circ 15'$, and is equal to $\{\log \sin 31^\circ 15' + 10\}$. For distinction $\log \sin 31^\circ 15'$ is read "log-sine" (short for logarithmic sine) whilst the tabular logarithm $L\sin 31^\circ 15'$ is read "el-sine."

As in the case of the use of tables of the natural trigonometrical functions, the form of setting out logarithmic work is important. The following general rules should be attended to.

(*a*) Arrange all work in strictly logical form: do not be slip-shod.

(*b*) Arrange logarithms to be added *in columns*, not in rows: keep the decimal points under one another.

(*c*) Do not write any unnecessary figures.

(*d*) Do all caloulations for interpolation on scrap-paper or set aside a column on the right-hand side of the page for the purpose.

(*e*) Keep the mantissae positive.

EXERCISES.

1. Find the value of $\sqrt[3]{\frac{3}{7}}$.

[*Model Solution.*]

$$\log \sqrt[3]{\tfrac{3}{7}} = \tfrac{1}{3}(\log 3 - \log 7)$$
$$= 0{\cdot}15904$$
$$\underline{-\ 0{\cdot}28170}$$
$$= \bar{1}{\cdot}87734 \quad \text{diff.} = 54$$

From the tables, $\log 7{\cdot}53 = \quad 680$ diff. $= 57$;
$4 \quad 737$

$$\therefore\ \sqrt[3]{\tfrac{3}{7}} = 7{\cdot}53\tfrac{54}{57} \times 10^{-1}$$
$$= \underline{0{\cdot}75395.}$$

2. Find the value of $(0{\cdot}0327)^3$.

[*Model Solution.*]

$$\log (0{\cdot}0327)^3 = 3\,(\log 0{\cdot}0327)$$
$$= 3\,(\bar{2}{\cdot}51455)$$
$$= \bar{5}{\cdot}54365 \quad \text{diff.} = 82$$

From the tables, $\log 3{\cdot}49 = \quad 283$ diff. $= 124$;
$50 = \quad 407$

$$\therefore\ (0{\cdot}0327)^3 = 3{\cdot}49\tfrac{82}{124} \times 10^{-5}$$
$$= \underline{{\cdot}000034966.}$$

3. Find the value of $a = \dfrac{24 \times \sin 72^\circ 4'}{\sin 65^\circ 59' 42''}$.

[*Model Solution.*]

$$\log a = \log 24 + L\sin 72^\circ 4' - L\sin 65^\circ 59' 42'' \quad (\alpha)$$

$$\begin{array}{rlrl}
= & 1{\cdot}38021 & -9{\cdot}96017 & (\delta) \\
 & 9{\cdot}97821 \ (\beta) & \underline{54} & (\epsilon) \\
 & \underline{16} \ (\gamma) & & \\
= & 11{\cdot}35858 & & \\
- & \underline{9{\cdot}96071} & & \\
= & 1{\cdot}39787 & &
\end{array}$$

From the tables

$$\left.\begin{array}{rr}
 & 1{\cdot}39787 \\
\log 2{\cdot}49 = & 620 \\
50 = & 795
\end{array}\right\}\ \begin{array}{l}\text{diff.} = 167 \\ \text{diff.} = 175\,;\end{array}$$

$$\therefore\ a = 2{\cdot}49\tfrac{167}{175} \times 10^1$$

$$= \underline{24{\cdot}995}.$$

4. Find A, given $\tan\dfrac{A}{2} = \sqrt{\dfrac{191{\cdot}42 \times 160{\cdot}83}{674{\cdot}10 \times 321{\cdot}85}}$.

[*Model Solution.*]

$$\log\left(\frac{191{\cdot}42 \times 160{\cdot}83}{674{\cdot}10 \times 321{\cdot}85}\right) \begin{array}{rlrl}
= & 2{\cdot}28103 \ (\zeta) & -2{\cdot}82866 & (\kappa) \\
 & 95 \ (\eta) & 6 & (\lambda) \\
 & 2{\cdot}20412 \ (\theta) & 2{\cdot}50651 & (\mu) \\
 & \underline{225} \ (\iota) & \underline{115} & (\nu) \\
= & 4{\cdot}48835 & & \\
- & \underline{5{\cdot}33638} & & \\
= & \bar{1}{\cdot}15197 & &
\end{array}$$

$$\therefore\ \log\tan\frac{A}{2} = \bar{1}{\cdot}57599\,;$$

$$\therefore\ L\tan\frac{A}{2} = 9{\cdot}57599$$

From the tables

$$\left.\begin{array}{rr}
L\tan\dfrac{A}{2} = & 9{\cdot}57599 \\
L\tan 20^\circ 30' = & 274 \\
40' = & 658
\end{array}\right\}\ \begin{array}{l}\text{diff.} = 325 \\ \text{diff. for } 10' = 384\,;\end{array}$$

$$\therefore\ \frac{A}{2} = 20^\circ 30' + \tfrac{325}{384} \times 10'$$

$$= 20^\circ 38' 28''\,;$$

$$\therefore\ A = \underline{41^\circ 16' 56''}.$$

Find the values of

5. $\dfrac{1}{(1{\cdot}44)^3}$. 8. $0{\cdot}81 \div \sqrt{112}$. 11. $(\frac{8}{7})^{\frac{1}{4}}$.

6. $\sqrt{0{\cdot}0125}$. 9. $\sqrt[5]{\frac{7}{80}}$. 12. $(0{\cdot}03)^5 \div (0{\cdot}12)^4$.

7. $\dfrac{(1{\cdot}08)^4}{(0{\cdot}0147)^3}$. 10. $(1{\cdot}05)^{20}$. 13. $(18)^{\sqrt{2}}$.

Find the logarithms of

14. $(\sin 18^\circ\, 37')^{-2}$. 16. $\sqrt[3]{(\tan 13^\circ\, 12'\, 45'')}$.

15. $(\tan 35^\circ\, 42')^3$. 17. $(\cos 26^\circ\, 33')^{-\frac{1}{4}}$.

Find the value of x, correct to three decimal places, that satisfies each of the equations:

18. $15^x = 20$. 19. $7^x = 3^{x+1} \div 2^{x-2}$.

20. If
$$\log \frac{a-b}{3} = \tfrac{1}{2}(\log a + \log b),$$
shew that
$$a^2 + b^2 = 11ab.$$

(α) The additional 10s in the tabular logarithms cancel one another.

(β) $L \sin 72^\circ$ from the tables. (γ) prop. diff. for $4'$.

(δ) $L \sin 65^\circ\, 50'$ from the tables. (ϵ) prop. diff. for $9'\, 42''$.

(ζ) log 191. (η) diff. for 42. (θ) log 160. (ι) diff. for 83.

(κ) log 674. (λ) diff. for 10. (μ) log 321. (ν) diff. for 85.

No.	0	1	2	3	4	5	6	7	8	9
100	00000	00043	00087	00130	00173	00217	00260	00303	00346	00389
1	00432	00475	00518	00560	00604	00647	00689	00732	00775	00817
2	00860	00903	00945	00988	01030	01072	01115	01157	01199	01242
3	01284	01326	01368	01410	01452	01494	01536	01578	01620	01662
4	01703	01745	01787	01828	01870	01912	01953	01995	02036	02078
5	02119	02160	02202	02243	02284	02325	02366	02408	02449	02490
6	02531	02572	02612	02653	02694	02735	02776	02816	02857	02898
7	02938	02979	03019	03060	03100	03141	03181	03222	03262	03302
8	03342	03383	03423	03463	03503	03543	03583	03623	03663	03703
9	03743	03782	03822	03862	03902	03941	03981	04021	04060	04100
11	04139	04532	04922	05308	05690	06070	06446	06819	07188	07555
2	07918	08279	08636	08991	09342	09691	10037	10380	10721	11060
3	11394	11727	12057	12385	12710	13033	13354	13672	13988	14301
4	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319
5	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140
6	20412	20710	20952	21219	21484	21748	22011	22272	22531	22789
7	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285
8	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646
9	27875	28103	28330	28556	28780	29003	29226	29447	29667	29885
20	30103	30320	30535	30750	30963	31175	31387	31597	31806	32015
1	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044
2	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984
3	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840
4	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620
5	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330
6	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975
7	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560
8	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090
9	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996
1	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379
2	50515	50651	50786	50920	51055	51188	51322	51455	51587	51720
3	51857	51983	52114	52244	52375	52504	52634	52763	52892	53020
4	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283
5	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509
6	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703
7	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864
8	57978	58093	58206	58320	58433	58546	58659	58771	58883	58995
9	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097
40	60206	60314	60423	60531	60638	60746	60853	60959	61066	61172
1	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221
2	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246
3	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246
4	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225
5	65321	65418	65514	65610	65706	65801	65896	65992	66087	66181
6	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117
7	67210	67302	67394	67486	67578	67609	67761	67852	67943	68034
8	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931
9	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672

One more figure

The "Rule of Proportional Parts" may be used to find two more figures

No.	0	1	2	3	4	5	6	7	8	9
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672
1	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517
2	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346
3	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159
4	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957
5	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741
6	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511
7	75587	75664	75740	75815	75891	75967	76042	76118	76193	76268
8	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012
9	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462
1	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169
2	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865
3	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550
4	80618	80686	80754	80821	80889	80956	81023	81090	81158	81224
5	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889
6	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543
7	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187
8	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822
9	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065
1	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673
2	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273
3	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864
4	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448
5	87506	87564	87622	87680	87737	87795	87852	87910	87967	88024
6	88081	88138	88196	88252	88309	88366	88423	88480	88536	88593
7	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154
8	89209	89265	89321	89376	89432	89487	89542	89597	89653	89708
9	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255
80	90309	90363	90417	90472	90526	90580	90633	90687	90741	90795
1	90849	90902	90956	91009	91062	91116	91169	91222	91275	91328
2	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855
3	91908	91960	92012	92065	92117	92169	92221	92273	92324	92376
4	92428	92480	92531	92583	92634	92686	92737	92788	92840	92891
5	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399
6	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902
7	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399
8	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890
9	94939	94988	95036	95085	95134	95182	95231	95274	95328	95376
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856
1	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332
2	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802
3	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267
4	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727
5	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182
6	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632
7	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078
8	99123	99167	99211	90255	99300	99344	99388	99432	99476	99520
9	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957

The "Rule of Proportional Parts" may be used to find two more figures.

Sines	0′	10′	20′	30′	40′	50′	60′	
0°	$-\infty$	7·46373	7·76475	7·94084	8·06578	8·16268	8·24186	89°
1	8·24186	8·30879	8·36678	8·41792	8·46366	8·50504	8·54282	88
2	8·54282	8·57757	8·60973	8·63968	8·66769	8·69400	8·71880	87
3	8·71880	8·74226	8·76451	8·78568	8·80585	8·82513	8·84358	86
4	8·84358	8·86128	8·87829	8·89494	8·91040	8·92561	8·94030	85
5	8·94030	8·95450	8·96825	8·98157	8·99450	9·00704	9·01923	84
6	9·01923	9·03109	9·04262	9·05386	9·06481	9·07548	9·08589	83
7	9·08589	9·09606	9·10599	9·11570	9·12519	9·13447	9·14356	82
8	9·14356	9·15245	9·16116	9·16970	9·17807	9·18628	9·19433	81
9	9·19433	9·20223	9·20999	9·21761	9·22509	9·23244	9·23967	80
10	9·23967	9·24677	9·25376	9·26063	9·26739	9·27405	9·28060	79
11	9·28060	9·28705	9·29340	9·29966	9·30582	9·31189	9·31788	78
12	9·31788	9·32378	9·32960	9·33534	9·34100	9·34658	9·35209	77
13	9·35209	9·35752	9·36289	9·36819	9·37341	9·37858	9·38368	76
14	9·38368	9·38871	9·39369	9·39860	9·40346	9·40825	9·41300	75
15	9·41300	9·41768	9·42232	9·42690	9·43143	9·43591	9·44034	74
16	9·44034	9·44472	9·44905	9·45334	9·45758	9·46178	9·46594	73
17	9·46594	9·47005	9·47411	9·47814	9·48213	9·48607	9·48998	72
18	9·48998	9·49385	9·49768	9·50148	9·50523	9·50896	9·51264	71
19	9·51264	9·51629	9·51991	9·52350	9·52705	9·53057	9·53405	70
20	9·53405	9·53751	9·54093	9·54433	9·54769	9 55102	9·55433	69
21	9·55433	9·55761	9·56805	9·56408	9·56727	9·57044	9·57358	68
22	9·57358	9·57669	9·57978	9·58284	9·58588	9·58889	9·59188	67
23	9·59188	9·59484	9·59788	9·60070	9·60360	9·60646	9·60931	66
24	9·60931	9·61214	9·61494	9·61773	9·62049	9·62323	9·62595	65
25	9·62595	9·62865	9·63133	9·63398	9·63662	9·63924	9·64184	64
26	9·64184	9·64442	9·64698	9·64953	9·65205	9·65456	9·65705	63
27	9·65705	9·65952	9·66197	9·66441	9·66682	9·66923	9·67161	62
28	9·67161	9·67398	9·67633	9·67867	9·68098	9·68328	9·68557	61
29	9·68557	9·68784	9·69010	9·69234	9·69456	9·69677	9·69897	60
30	9·69897	9·70115	9·70332	9·70547	9·70761	9·70973	9·71184	59
31	9·71184	9·71393	9·71602	9·71809	9·72014	9·72218	9·72421	58
32	9·72421	9·72622	9·72823	9·73022	9·73219	9·73416	9·73611	57
33	9·73611	9·73805	9·73997	9·74189	9·74379	9·74568	9·74756	56
34	9·74756	9·74943	9·75128	9·75313	9·75496	9·75678	9·75859	55
35	9·75859	9·76039	9·76218	9·76395	9·76572	9·76747	9·76922	54
36	9·76922	9·77095	9·77268	9·77439	9·77609	9·77778	9·77946	53
37	9·77946	9·78113	9·78280	9·78445	9·78609	9·78772	9·78934	52
38	9·78934	9·79095	9·79256	9·79415	9·79573	9·79731	9·79887	51
39	9·79887	9·80043	9·80197	9·80351	9·80504	9·80656	9·80807	50
40	9·80807	9·80957	9·81106	9·81254	9·81402	9·81549	9·81694	49
41	9·81694	9·81839	9·81983	9·82126	9·82269	9·82410	9·82551	48
42	9·82551	9·82691	9·82830	9·82968	9·83106	9·83242	9·83378	47
43	9·83378	9·83513	9·83648	9·83781	9·83914	9·84046	9·84177	46
44	9·84177	9·84308	9·84438	9·84566	9·84694	9·84822	9·84949	45
	60′	50′	40′	30′	20′	10′	0′	Co-sines

Sines	0′	10′	20′	30′	40′	50′	60′	
45°	9·84949	9·85074	9·85200	9·85324	9·85448	9·85571	9·85693	44°
46	9·85693	9·85815	9·85936	9·86056	9·86176	9·86295	9·86413	43
47	9·86413	9·86530	9·86647	9·86763	9·86879	9·86993	9·87107	42
48	9·87107	9·87221	9·87334	9·87446	9·87557	9·87668	9·87778	41
49	9·87778	9·87887	9·87996	9·88105	9·88212	9·88319	9·88425	40
50	9·88425	9·88531	9·88636	9·88741	9·88844	9·88948	9·89050	39
51	9·89050	9·89152	9·89254	9·89354	9·89455	9·89554	9·89653	38
52	9·89653	9·89752	9·89849	9·89947	9·90043	9·90139	9·90235	37
53	9·90235	9·90330	9·90424	9·90518	9·90611	9·90704	9·90796	36
54	9·90796	9·90887	9·90978	9·91069	9·91158	9·91248	9·91336	35
55	8·91336	9·91425	9·91512	9·91599	9·91686	9·91772	9·91857	34
56	9·91857	9·91942	9·92027	9·92111	9·92194	9·92277	9·92359	33
57	9·92359	9·92441	9·92522	9·92603	9·92683	9·92763	9·92842	32
58	9·92842	9·92921	9·92999	9·93077	9·93154	9·93230	9·93307	31
59	9·93307	9·93382	9·93457	9·93532	9·93606	9·93680	9·93753	30
60	9·93753	9·93826	9·93898	9·93970	9·94041	9·94112	9·94182	29
61	9·94182	9·94252	9·94321	9·94390	9·94458	9·94526	9·94593	28
62	9·94593	9·94660	9·94727	9·94793	9·94858	9·94923	9·94988	27
63	9·94988	9·95052	9·95116	9·95179	9·95242	9·95304	9·95366	26
64	9·95366	9·95427	9·95488	9·95549	9·95609	9·95668	9·95728	25
65	9·95728	9·95786	9·95845	9·95902	9·95960	9·96017	9·96073	24
66	9·96073	9·96129	9·96185	9·96240	9·96294	9·96349	9·96403	23
67	9·96403	9·96456	9·96509	9·96562	9·96614	9·96665	9·96717	22
68	9·96717	9·96769	9·96818	9·96868	9·96917	9·96966	9·97015	21
69	9·97015	9·97063	9·97111	9·97159	9·97206	9·97252	9·97299	20
70	4·97299	9·97344	9·97390	9·97435	9·97479	9·97523	9·97567	19
71	9·97567	9·87610	9·97653	9·97696	9·97738	9·97779	9·97821	18
72	9·97821	9·97861	9·97902	9·97942	9·97982	9·98021	9·98060	17
73	9·98060	9·98098	9·98136	9·98174	9·98211	9·98248	9·98284	16
74	9·98284	9·98320	9·98356	9·98391	9·98426	9·98460	9·98494	15
75	9·98494	9·98528	9·98561	9·98594	9·98627	9·98659	9·98690	14
76	9·98690	9·98722	9·98753	9·98783	9·98813	9·98843	9·98872	13
77	9·98872	9·98901	9·98930	9·98958	9·98986	9·99013	9·99040	12
78	9·99040	9·99067	9·99093	9·99119	9·99145	9·99170	9·99195	11
79	9·99195	9·99219	9·99243	9·99267	9·99290	9·99313	9·99335	10
80	9·99335	9·99357	9·99379	9·99400	9·99421	9·99442	9·99462	9
81	9·99462	9·99482	9·99501	9·99520	9·99539	9·99557	9·99575	8
82	9·99575	9·99593	9·99610	9·99627	9·99643	9·99659	9·99675	7
83	9·99675	9·99690	9·99705	9·99720	9·99734	9·99748	9·99761	6
84	9·99761	9·99775	9·99787	9·99800	9·99812	9·99823	9·99834	5
85	9·99834	9·99845	9·99856	9·99866	9·99876	9·99885	9·99894	4
86	9·99894	9·99903	9·99911	9·99919	9·99926	9·99934	9·99940	3
87	9·99940	9·99947	9·99953	9·99959	9·99964	9·99969	9·99974	2
88	9·99974	9·99978	9·99982	9·99985	9·99988	9·99991	9·99993	1
89	9·99993	9·99995	9·99997	9·99998	9·99999	9·999998	10·00000	0
	60′	50′	40′	30′	20′	10′	0′	Co-sines

Tangents	0′	10′	20′	30′	40′	50′	60′	
0°	−∞	7·46373	7·76476	7·94086	8·06581	8·16273	8·24192	89°
1	8·24192	8·30888	8·36689	8·41807	8·46385	8·50527	8·54308	88
2	8·54308	8·57788	8·61009	8·64009	8·66818	8·69453	8·71940	87
3	8·71940	8·74292	8·76525	8·78649	8·80674	8·82610	8·84464	86
4	8·84464	8·86243	8·87953	8·89598	8·91185	8·92716	8·94195	85
5	8·94195	8·95627	8·97013	8·98358	8·99662	9·00930	9·02162	84
6	9·02162	9·03361	9·04528	9·05666	9·06775	9·07858	9·08914	83
7	9·08914	9·09947	9·10956	9·11943	9·12909	9·13854	9·14780	82
8	9·14780	9·15688	9·16577	9·17450	9·18306	9·19146	9·19971	81
9	9·19971	9·20782	9·21578	9·22361	9·23130	9·23887	9·24632	80
10	9·24632	9·25365	9·26086	9·26797	9·27496	9·28186	9·28865	79
11	9·28865	9·29535	9·30195	9·30846	9·31489	9·32122	9·32747	78
12	9·32747	9·33365	9·33974	9·34576	9·35170	9·35757	9·36336	77
13	9·36336	9·36909	9·37476	9·38035	9·38589	9·39136	9·39677	76
14	9·39677	9·40212	9·40742	9·41266	9·41784	9·42297	9·42805	75
15	9·42805	9·43308	9·43806	9·44299	9·44787	9·45271	9·45750	74
16	9·45750	9·46224	9·46694	9·47160	9·47622	9·48080	9·48534	73
17	9·48534	9·48984	9·49430	9·49872	9·50311	9·50746	9·51178	72
18	9·51178	9·51606	9·52031	9·52452	9·52870	9·53285	9·53697	71
19	9·53697	9·54106	9·54512	9·54915	9·55315	9·55712	9·56107	70
20	9·56107	9·56458	9·56887	9·57274	9·57658	9·58039	9·58418	69
21	9·58418	9·58794	9·59168	9·59540	9·59909	9·60276	9·60641	68
22	9·60641	9·61004	9·61364	9·61722	9·62079	9·62433	9·62785	67
23	9·62785	9·63135	9·63484	9·63830	9·64175	9·64517	9·64858	66
24	9·64858	9·65197	9·65535	9·65870	9·66204	9·66537	9·66867	65
25	9·66867	9·67196	9·67524	9·67850	9·68174	9·68497	9·68818	64
26	9·68818	9·69138	9·69457	9·69774	9·70089	9·70404	9·70717	63
27	9·70717	9·71028	9·71339	9·71648	9·71955	9·72262	9·72567	62
28	9·72567	9·72872	9·73175	9·73476	9·73777	9·74077	9·74375	61
29	9·74375	9·74673	9·74969	9·75264	9·75558	9·75852	9·76144	60
30	9·76144	9·76435	9·76726	9·77015	9·77303	9·77591	9·77877	59
31	9·77877	9·78163	9·78448	9·78732	9·79015	9·79297	9·79579	58
32	9·79579	9·79860	9·80140	9·80419	9·80697	9·80975	9·81252	57
33	9·81252	9·81528	9·81803	9·82078	9·82352	9·82626	9·82899	56
34	9·82899	9·83171	9·83442	9·83713	9·83984	9·84254	9·84523	55
35	9·84523	9·84791	9·85059	9·85327	9·85594	9·85860	9·86126	54
36	9·86126	9·86392	9·86656	9·86921	9·87185	9·87448	9·87711	53
37	9·87711	9·87974	9·88236	9·88498	9·88759	9·89020	9·89281	52
38	9·89281	9·89541	9·89801	9·90061	9·90320	9·90578	9·90837	51
39	9·90837	9·91095	9·91353	9·91610	9·91868	9·92125	9·92381	50
40	9·92381	9·92638	9·92894	9·93150	9·93406	9·93661	9·93916	49
41	9·93916	9·94171	9·94426	9·94681	9·94935	9·95189	9·95444	48
42	9·95444	9·95698	9·95952	9·96205	9·96459	9·96712	9·96966	47
43	9·96966	9·97219	9·97472	9·97725	9·97978	9·98231	9·98484	46
44	9·98434	9·98737	9·98989	9·99242	9·99495	9·99747	10·00000	45
	60′	50′	40′	30′	20′	10′	0′	Cotangents

Tangents	0′	10′	20′	30′	40′	50′	60′	
45°	10·00000	10·00253	10·00505	10·00758	10·01011	10·01263	10·01516	**44°**
46	10·01516	10·01769	10·02022	10·02275	10·02528	10·02781	10·03034	**43**
47	10·03034	10·03288	10·03541	10·03795	10·04048	10·04302	10·04556	**42**
48	10·04556	10·04810	10·05065	10·05319	10·05574	10·05829	10·06084	**41**
49	10·06084	10·06339	10·06594	10·06850	10·07106	10·07362	10·07619	**40**
50	10·07619	10·07875	10·08132	10·08390	10·08647	10·08905	10·09163	**39**
51	10·09163	10·09422	10·09680	10·09939	10·10199	10·10459	10·10719	**38**
52	10·10719	10·10980	10·11241	10·11502	10·11764	10·12026	10·12289	**37**
53	10·12289	10·12552	10·12815	10·13079	10·13344	10·13608	10·13874	**36**
54	10·13874	10·14140	10·14406	10·14673	10·14941	10·15209	10·15477	**35**
55	10·15477	10·15746	10·16016	10·16287	10·16558	10·16829	10·17101	**34**
56	10·17101	10·17374	10·17648	10·17922	10·18197	10·18472	10·18748	**33**
57	10·18748	10·19025	10·19303	10·19581	10·19860	10·20140	10·20421	**32**
58	10·20421	10·20703	10·20985	10·21268	10·21552	10·21837	10·22123	**31**
59	10·22123	10·22409	10·22697	10·22985	10·23275	10·23565	10·23856	**30**
60	10·23856	10·24148	10·24442	10·24736	10·25031	10·25327	10·25625	**29**
61	10·25625	10·25923	10·26223	10·26524	10·26825	10·27128	10·27433	**28**
62	10·27433	10·27738	10·28045	10·28352	10·28661	10·28972	10·29283	**27**
63	10·29283	10·29596	10·29911	10·30226	10·30543	10·30862	10·31182	**26**
64	10·31182	10·31503	10·31826	10·32150	10·32476	10·32804	10·33133	**25**
65	10·33133	10·33463	10·33796	10·34130	10·34465	10·34803	10·35142	**24**
66	10·35142	10·35483	10·35825	10·36170	10·36516	10·36865	10·37215	**23**
67	10·37215	10·37567	10·37921	10·38278	10·38636	10·38996	10·39359	**22**
68	10·39359	10·39724	10·40091	10·40460	10·40832	10·41206	10·41582	**21**
69	10·41582	10·41961	10·42342	10·42726	10·43113	10·43502	10·43893	**20**
70	10·43893	10·44288	10·44685	10·45085	10·45488	10·45894	10·46303	**19**
71	10·46303	10·46715	10·47130	10·47548	10·47969	10·48394	10·48822	**18**
72	10·48822	10·49254	10·49689	10·50128	10·50570	10·51016	10·51466	**17**
73	10·51466	10·51920	10·52378	10·52840	10·53306	10·53776	10·54250	**16**
74	10·54250	10·54729	10·55213	10·55701	10·56194	10·56692	10·57195	**15**
75	10·57195	10·57703	10·58216	10·58734	10·59258	10·59788	10·60323	**14**
76	10·60323	10·60864	10·61411	10·61965	10·62524	10·63091	10·63664	**13**
77	10·63664	10·64243	10·64830	10·65424	10·66026	10·66635	10·67253	**12**
78	10·67253	10·67878	10·68511	10·69154	10·69805	10·70465	10·71135	**11**
79	10·71135	10·71814	10·72504	10·73203	10·73914	10·74635	10·75368	**10**
80	10·75368	10·76113	10·76870	10·77639	10·78422	10·79218	10·80029	**9**
81	10·80029	10·80854	10·81694	10·82550	10·83423	10·84312	10·85220	**8**
82	10·85220	10·86146	10·87091	10·88057	10·89044	10·90053	10·91086	**7**
83	10·91086	10·92142	10·93225	10·94334	10·95472	10·96639	10·97838	**6**
84	10·97838	10·99070	11·00338	11·01642	11·02987	11·04373	11·05805	**5**
85	11·05805	11·07284	11·08815	11·10402	11·12047	11·13757	11·15536	**4**
86	11·15536	11·17390	11·19326	11·21351	11·23475	11·25708	11·28060	**3**
87	11·28060	11·30547	11·33184	11·35991	11·38991	11·42212	11·45692	**2**
88	11·45692	11·49473	11·53615	11·58193	11·63311	11·69112	11·75808	**1**
89	11·75808	11·83727	11·93419	12·05914	12·23524	12·53627	∞	**0**
	60′	50′	40′	30′	20′	10′	0′	Cotangents

Secants	0′	10′	20′	30′	40′	50′	60′	
0°	10·00000	10·000001	10·00001	10·00002	10·00003	10·00005	10·00007	89°
1	10·00007	10·00009	10·00012	10·00015	10·00018	10·00022	10·00026	88
2	10·00026	10·00031	10·00036	10·00041	10·00047	10·00053	10·00060	87
3	10·00060	10·00066	10·00074	10·00081	10·00089	10·00097	10·00106	86
4	10·00106	10·00115	10·00124	10·00134	10·00144	10·00155	10·00166	85
5	10·00166	10·00177	10·00188	10·00200	10·00213	10·00225	10·00239	84
6	10·00239	10·00252	10·00266	10·00280	10·00295	10·00310	10·00325	83
7	10·00325	10·00341	10·00357	10·00373	10·00390	10·00407	10·00425	82
8	10·00425	10·00443	10·00461	10·00480	10·00499	10·00518	10·00538	81
9	10·00538	10·00558	10·00579	10·00600	10·00621	10·00643	10·00665	80
10	10·00665	10·00687	10·00710	10·00733	10·00757	10·00781	10·00805	79
11	10·00805	10·00830	10·00855	10·00881	10·00907	10·00933	10·00960	78
12	10·00960	10·00987	10·01014	10·01041	10·01070	10·01099	10·01128	77
13	10·01128	10·01157	10·01187	10·01217	10·01247	10·01278	10·01310	76
14	*10·01310*	*10·01341*	*10·01373*	*10·01406*	*10·01439*	*10·01472*	*10·01506*	75
15	10·01506	10·01540	10·01574	10·01609	10·01644	10·01680	10·01716	74
16	10·01716	10·01752	10·01789	10·01826	10·01864	10·01902	10·01940	73
17	10·01940	10·01979	10·02018	10·02058	10·02098	10·02139	10·02179	72
18	10·02179	10·02221	10·02262	10·02304	10·02347	10·02390	10·02433	71
19	10·02433	10·02477	10·02521	10·02565	10·02610	10·02656	10·02701	70
20	10·02701	10·02748	10·02794	10·02841	10·02889	10·02937	10·02985	69
21	10·02985	10·03034	10·03083	10·03132	10·03182	10·03233	10·03283	68
22	10·03283	10·03335	10·03386	10·03438	10·03491	10·03544	10·03597	67
23	10·03597	10·03651	10·03706	10·03760	10·03815	10·03871	10·03927	66
24	10·03927	10·03983	10·04040	10·04098	10·04156	10·04214	10·04272	65
25	10·04272	10·04332	10·04391	10·04451	10·04511	10·04573	10·04634	64
26	10·04634	10·04696	10·04758	10·04821	10·04884	10·04948	10·05012	63
27	10·05012	10·05077	10·05142	10·05207	10·05273	10·05340	10·05407	62
28	*10·05407*	*10·05474*	*10·05542*	*10·05610*	*10·05679*	*10·05748*	*10·05818*	61
29	10·05818	10·05888	10·05959	10·06030	10·06102	10·06174	10·06247	60
30	10·06247	10·06320	10·06394	10·06468	10·06543	10·06618	10·06693	59
31	10·06693	10·06770	10·06846	10·06923	10·07001	10·07079	10·07158	58
32	10·07158	10·07237	10·07317	10·07397	10·07478	10·07559	10·07641	57
33	10·07641	10·07723	10·07806	10·07889	10·07973	10·08058	10·08143	56
34	10·08143	10·08228	10·08314	10·08401	10·08488	10·08575	10·08664	55
35	10·08664	10·08752	10·08842	10·08931	10·09022	10·09113	10·09204	54
36	10·09204	10·09296	10·09389	10·09482	10·09576	10·09670	10·09765	53
37	10·09765	10·09861	10·09957	10·10053	10·10151	10·10248	10·10347	52
38	10·10347	10·10446	10·10545	10·10646	10·10746	10·10848	10·10950	51
39	10·10950	10·11052	10·11156	10·11259	10·11364	10·11469	10·11575	50
40	10·11575	10·11681	10·11788	10·11895	10·12004	10·12113	10·12222	49
41	10·12222	10·12332	10·12443	10·12554	10·12666	10·12779	10·12893	48
42	*10·12893*	*10·13007*	*10·13121*	*10·13237*	*10·13353*	*10·13470*	*10·13587*	47
43	10·13587	10·13705	10·13824	10·13944	10·14064	10·14185	10·14307	46
44	10·14307	10·14429	10·14552	10·14676	10·14800	10·14926	10·15052	45
	60′	50′	40′	30′	20′	10′	0′	Cosecants

Secants	0′	10′	20′	30′	40′	50′	60′	
45°	10·15052	10·15178	10·15306	10·15434	10·15563	10·15692	10·15823	44°
46	10·15823	10·15954	10·16086	10·16219	10·16352	10·16487	10·16622	43
47	10·16623	10·16758	10·16894	10·17032	10·17170	10·17309	10·17449	42
48	10·17449	10·17590	10·17731	10·17874	10·18017	10·18161	10·18306	41
49	10·18306	10·18451	10·18598	10·18746	10·18894	10·19043	10·19193	40
50	10·19193	10·19344	10·19496	10·19649	10·19803	10·19957	10·20113	39
51	10·20113	10·20269	10·20427	10·20585	10·20744	10·20905	10·21066	38
52	10·21066	10·21228	10·21391	10·21555	10·21720	10·21887	10·22054	37
53	10·22054	10·22222	10·22391	10·22561	10·22732	10·22905	10·23078	36
54	10·23078	10·23253	10·23428	10·23605	10·23782	10·23961	10·24141	35
55	10·24141	10·24322	10·24504	10·24687	10·24872	10·25057	10·25243	34
56	10·25243	10·25432	10·25621	10·25811	10·26003	10·26195	10·26389	33
57	10·26389	10·26584	10·26781	10·26978	10·27177	10·27378	10·27579	32
58	10·27579	10·27782	10·27986	10·28191	10·28398	10·28607	10·28816	31
59	10·28816	10·29027	10·29239	10·29453	10·29668	10·29885	10·30103	30
60	10·30103	10·30323	10·30544	10·30766	10·30990	10·31216	10·31443	29
61	10·31443	10·31672	10·31902	10·32134	10·32367	10·32602	10·32839	28
62	10·32839	10·33078	10·33318	10·33559	10·33803	10·34048	10·34295	27
63	10·34295	10·34544	10·34795	10·35047	10·35302	10·35557	10·35816	26
64	10·35816	10·36076	10·36338	10·36602	10·36867	10·37135	10·37405	25
65	10·37405	10·37677	10·37951	10·38227	10·38506	10·38786	10·39069	24
66	10·39069	10·39354	10·39641	10·39930	10·40222	10·40516	10·40812	23
67	10·40812	10·41111	10·41412	10·41716	10·42022	10·42331	10·42642	22
68	10·42642	10·42956	10·43273	10·43592	10·43915	10·44239	10·44567	21
69	10·44567	10·44898	10·45231	10·45567	10·45907	10·46249	10·46595	20
70	10·46595	10·46944	10·47295	10·47650	10·48009	10·48371	10·48736	19
71	10·48736	10·49104	10·49477	10·49852	10·50232	10·50615	10·51002	18
72	10·51002	10·51393	10·51787	10·52186	10·52589	10·52995	10·53406	17
73	10·53406	10·53822	10·54242	10·54666	10·55095	10·55528	10·55966	16
74	10·55966	10·56409	10·56857	10·57310	10·57768	10·58232	10·58700	15
75	10·58700	10·59175	10·59654	10·60140	10·60631	10·61129	10·61632	14
76	10·61632	10·62142	10·62659	10·63181	10·63711	10·64248	10·64791	13
77	10·64791	10·65342	10·65900	10·66466	10·67040	10·67622	10·68212	12
78	10·68212	10·68811	10·69418	10·70034	10·70660	10·71295	10·71940	11
79	10·71940	10·72595	10·73261	10·73937	10·74624	10·75323	10·76033	10
80	10·76033	10·76756	10·77491	10·78239	10·79001	10·79777	10·80567	9
81	10·80567	10·81372	10·82193	10·83030	10·83884	10·84755	10·85644	8
82	10·85644	10·86553	10·87481	10·88430	10·89401	10·90394	10·91411	7
83	10·91411	10·92452	10·93519	10·94614	10·95738	10·96891	10·98077	6
84	10·98077	10·99296	11·00550	11·01843	11·03175	11·04550	11·05970	5
85	11·05970	11·07439	11·08960	11·10536	11·12171	11·13872	11·15642	4
86	11·15642	11·17487	11·19415	11·21432	11·23549	11·25774	11·28120	3
87	11·28120	11·30600	11·33231	11·36032	11·39027	11·42243	11·45718	2
88	11·45718	11·49496	11·53634	11·58208	11·63322	11·69121	11·75814	1
89	11·75814	11·83732	11·93422	12·05916	12·23525	12·53627	∞	0
	60′	50′	40′	30′	20′	10′	0′	Cosecants

§ 16. Discrepancies due to irregularity and insensibility.

In § 7, when discussing the use of Tables of Natural Sines, Cosines, etc., it was stated that roughly four-, five-, and seven-figure Tables could be trusted to give results correct to four, five and seven significant figures.

It is most important, in any calculation in which tables are used, to know to what extent they can be trusted to give correct results.

Discrepancies may arise in interpolation from one of two causes,

(i) **Irregularity**; i.e., there is a rapid change in the magnitude of consecutive differences.

(ii) **Insensibility**; i.e., the differences are too small.

If a graph of a particular set of tables is drawn, irregularity *is shown by a rapid change of slope of the* corresponding part of the curve; whilst insensibility is shown by the curve becoming parallel to the x-axis.

Thus in a table of natural sines we find

I.

Angle	0°	0° 10′	0° 20′	0° 30′	0° 40′	0° 50′	1° 0′
Sine	·00000	·00291	·00582	·00873	·01164	·01454	·01745
Diff.	291	291	291	291	290	290	

II.

Angle	89° 0′	89° 10′	89° 20′	89° 30′	89° 40′	89° 50′	90°
Sine	·99985	·99989	·99993	·99996	·99998	·99999	1·00000
Diff.	4	4	3	2	1	1	

Thus in the early part of the tables, the differences are regular and fairly sensible: a difference of 290 in the sine corresponds to a difference of 10′ in the angle, and thus the tables can be trusted to give results true to about 2″ or 3″ of angle.

On the other hand when the angle is nearly a right angle, the differences are not only rather irregular but very small; as a difference of 4 in the sine corresponds to a difference of 10′ in the angle, the tables cannot be trusted to give results true to even 2′ of angle, and are thus practically useless for this range. However, this difficulty can as a rule be avoided by using a different formula, involving one of the other trigonometrical ratios; for irregularity and insensibility occur for different ranges in different tables.

Thus for cosines the values for angles nearly 90° are regular and sensible, whilst for very small angles there is some irregularity, and great insensibility.

1. Examine the tables of tangents, cotangents, secants and cosecants, for irregularity and insensibility.

2. State for what ranges, for each trigonometrical ratio, the tables can be relied upon if we wish to secure results correct to one-tenth of a minute.

3. Find, from the five-figure tables on pp. 36—41, the values of

(1) sin 2° 13′ 27″, sin 6° 27′ 15″, cos 79° 12′ 20″, cos 87° 34′ 49″;

(2) tan 60° 21′ 45″, tan 79° 31′ 2″, cot 43° 2′ 40″, cot 4° 37′ 37″;

(3) sec 54° 54′ 54″, sec 82° 29′ 42″, cosec 31° 42′ 29″, cosec 6° 52′ 16″;

and compare the results obtained with the seven-figure results given in the answers.

The tables of logarithms to the base 10, or **common logarithms** as they are generally called, given on pp. 88, 89, shew that there is some irregularity for numbers between 1·0000 and 1·1000.

Thus

$$\left.\begin{array}{l}\log 1{\cdot}0000 = 0{\cdot}00000\\ \log 1{\cdot}0100 = 0{\cdot}00432\end{array}\right\}\text{ diff.} = 432$$
$$\left.\begin{array}{l}\log 1{\cdot}0100 = 0{\cdot}00432\\ \log 1{\cdot}0200 = 0{\cdot}00860\end{array}\right\}\text{ diff.} = 428$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\left.\begin{array}{l}\log 1{\cdot}0800 = 0{\cdot}03342\\ \log 1{\cdot}0900 = 0{\cdot}03743\end{array}\right\}\text{ diff.} = 401$$
$$\left.\begin{array}{l}\log 1{\cdot}0900 = 0{\cdot}03743\\ \log 1{\cdot}1000 = 0{\cdot}04139\end{array}\right\}\text{ diff.} = 396.$$

Except for this irregularity, the differences are quite large enough to give by interpolation two more significant figures instead of the zeros for a number between any two of the above.

For example, if it is required to find log 1·0137, we get by interpolation

$$\begin{aligned}\log 1{\cdot}0137 &= 0{\cdot}00432\\ &+ \quad\;\; 158\\ &= 0{\cdot}00590\end{aligned}$$

whilst the correct value is 0·00591.

Hence, as it is evident that the irregularity becomes greater, the nearer the logarithm approaches 1·0000, the tables have been extended for differences of only 0·0010 between 1·0000 and 1·1000, leaving only one more significant figure to be found by interpolation.

4. Find, from the five-figure tables on pp. 88, 89, the values of

    log 100·57, log 1000·3, log 0·102176, og 103·752,

    log 0·0108276, log 1·00002, log 1098·733,

and compare the results with the seven-figure results in the answers.

At the other end of the table the differences are very regular, but are considerably smaller.

Thus

$$\begin{array}{ll}
\log 8{\cdot}9000 = 0{\cdot}94939 & \\
 & \text{diff.} = 49 \\
\log 8{\cdot}9100 = 0{\cdot}94988 & \\
 & \text{diff.} = 48 \\
\log 8{\cdot}9200 = 0{\cdot}95036 & \\
 & \text{diff.} = 49 \\
\log 8{\cdot}9300 = 0{\cdot}95085 & \\
\dots\dots\dots\dots\dots\dots & \\
\log 8{\cdot}9700 = 0{\cdot}95279 & \\
 & \text{diff.} = 49 \\
\log 8{\cdot}9800 = 0{\cdot}95328 & \\
 & \text{diff.} = 48 \\
\log 8{\cdot}9900 = 0{\cdot}95376 & \\
 & \text{diff.} = 48. \\
\log 9{\cdot}0000 = 0{\cdot}95424 &
\end{array}$$

Hence, interpolation will give two more significant figures correctly; but, since the difference for 0·0100 is only 48, the difference for 0·0001 is roughly only $\frac{1}{2}$. Therefore we shall have two, or perhaps three in some cases, consecutive numbers with the same logarithms; so that although the logarithms of numbers with five significant figures can be found correctly by interpolation, yet if the logarithm is given, and the corresponding number has to be found by interpolation, there may be a mistake of 1 (or in rare cases, 2) in the fifth significant figure. But this is only an error of at most 2 in 80,000, i.e. considerably less than 1 in 10,000: thus the table is *uniformly* correct to ·01 °/ₒ.

5. Find, from the five-figure tables on pp. 88, 89, the values of

(1) log 899·72, log 892·6, log 0·723457, log 9·82679;

(2) *antilog 2·98764, antilog 3·95045, antilog $\bar{1}{\cdot}95399$;

comparing the results with the seven-figure results in the answers, and in (2) working out the error per cent.

---

* "Antilog" is the inverse notation for "the number whose logarithm is."

Tables of Tabular Logarithms of the Trigonometrical tables are similarly subject to irregularity and insensibility.

Too much reliance should not be placed on results obtained by interpolation,

(1) owing to *irregularity*, on the values of $L\sin$, $L\cos$, $L\tan$ of angles

between 0° and 5°, and 85° and 90°:

(2) owing to *insensibility*, on the values of $\sin^{-1}$, $\cos^{-1}$, obtained from

$L\sin$ of angles between 85° and 90°,

$L\cos$ of angles between 0° and 5°.

For the generality of angles between 5° and 85°, the differences for $10'$ of angle as given in the tables lie between 20 and 600: hence the tables may be trusted to give results correct to varying amounts from $\frac{1}{2}''$ to $1''$. In working problems the student should always notice and record the possible error in his results, remembering that insensibility only affects them one way, whilst irregularity affects them both ways. Thus $L\cos 6^\circ\, 42'\, 37''$ can be found correct to five places, from the tables on pp. 90, 91, to be $9{\cdot}99701$: whereas if it is given that $L\cos x = 9{\cdot}99701$, x may lie anywhere between $6^\circ\, 42'\, 36''$ and $6^\circ\, 13'\, 16''$.

When the possible error recorded is due to insensibility recourse must be had to tables calculated to a larger number of significant figures, if the possible error exceeds what is allowable on account of the nature of the problem: if the error is due to irregularity, tables calculated for smaller intervals must be used.

§ 17. Formulae adapted to Logarithms.

It has been shewn in § 15, that formulae which mainly consist of products, quotients, roots, and powers, are specially adapted to logarithmic computation. Of the formulae obtained in §§ 12, 13, viz.:

(i) $$\Delta = \tfrac{1}{2} bc \sin \mathsf{A},$$

(ii) $$\frac{a}{\sin \mathsf{A}} = \frac{b}{\sin \mathsf{B}} = \frac{c}{\sin \mathsf{C}},$$

(iii) $$\cos \mathsf{A} = \frac{b^2 + c^2 - a^2}{2bc},$$

only (i) and (ii) are specially adapted to logarithms.

By means of (i) the Area of a triangle can be calculated when any *two sides and the included angle are given.*

By combining (i) and (ii) a formula can be obtained which gives the area when *one side and two* (*i.e. three*) *angles* are given.

Thus,

$$\Delta = \tfrac{1}{2} bc \sin \mathsf{A}$$

$$= \frac{c^2}{2} \cdot \frac{\sin \mathsf{A} \sin \mathsf{B}}{\sin \mathsf{C}} \quad \ldots\ldots\ldots\ldots\ldots(\text{iv}).$$

If *two sides and an angle* (not the included angle) are given—say A, a, b—it is necessary to find the value or values of B by formula (ii), and thence the value or values of C and finally use formula (i).

If *three sides* are given, a formula can be found by eliminating **A** from formulae (ii) and (iii), thus :—

$$2bc\cos \mathsf{A} = b^2 + c^2 - a^2,$$

$$2bc\sin \mathsf{A} = 4\Delta.$$

Squaring and adding, since $\cos^2 \mathsf{A} + \sin^2 \mathsf{A} = 1$,

$$\therefore\ 4b^2c^2 = (b^2 + c^2 - a^2)^2 + 16\Delta^2,$$

$$\therefore\ 16\Delta^2 = (b^2 + c^2 - a^2)^2 - (2bc)^2$$

$$= (a + b + c)(b + c - a)(c + a - b)(a + b - c).$$

Let $2s = (a + b + c)$, so that s is the *Semiperimeter* of the triangle, then

$$\Delta = \sqrt{s\,.\,(s - a)(s - b)\,.\,(s - c)} \quad \text{.........(v)}.$$

The expressions s, $(s - a)$, $(s - b)$, $(s - c)$ are very useful and important. They are the lengths of certain lines connected with the triangle and the four circles, which touch the three sides, the inscribed circle and the three escribed circles.

69. Let **ABC** be a triangle, **DEF** the inscribed circle, let **AF, AE** be x units, **BF, BD,** y units, **CD, CE,** z units of length respectively.

Shew that

$$x + y + z = s,$$

and hence
$$x = s - a,$$
$$y = s - b,$$
$$z = s - c.$$

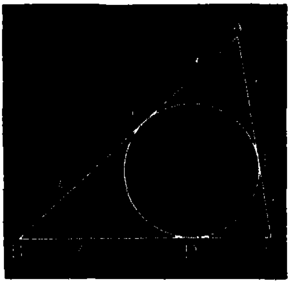


FIG. 49.

70. Let ABC be a triangle, D_1, E_1, F_1 the escribed circle which touches BC and the other two sides produced.

Shew that

$$AF_1 = AE_1 = s,$$
$$BF_1 = BD_1 = s - c,$$
$$CE_1 = CD_1 = s - b.$$

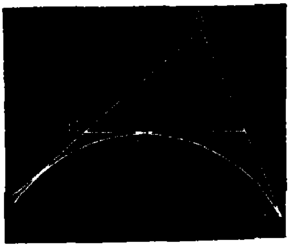


FIG. 50.

71. Let ABC be a triangle, I the centre of the inscribed circle DEF, r the radius of the circle.

Shew that

$$\Delta = \tfrac{1}{2} a \cdot r + \tfrac{1}{2} b \cdot r + \tfrac{1}{2} c \cdot r,$$

and hence

$$r = \frac{\Delta}{s}.$$

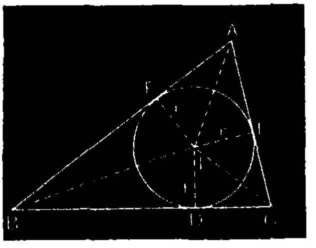


FIG. 51.

72. Let ABC be a triangle, I_1 the centre of the escribed circle $D_1E_1F_1$, which touches BC and the other two sides produced, r_1 the radius of the circle.

Shew that

$$\Delta = \tfrac{1}{2} b \cdot r_1 + \tfrac{1}{2} c \cdot r_1 - \tfrac{1}{2} a r_1,$$

and hence

$$r_1 = \frac{\Delta}{s - a}.$$

Similarly

$$r_2 = \frac{\Delta}{s - b}, \quad r_3 = \frac{\Delta}{s - c}.$$

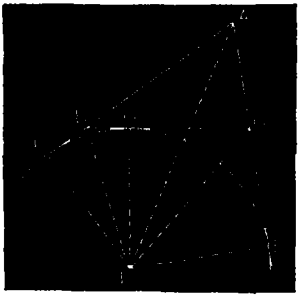


FIG. 52.

73. Let ABC be a triangle, I, r, the centre and radius of the inscribed circle DEF, I_1, r_1 the centre and radius of the escribed circle $D_1E_1F_1$ which touches BC and the other two sides produced. Join I_1D_1, I_1B, ID, IB.

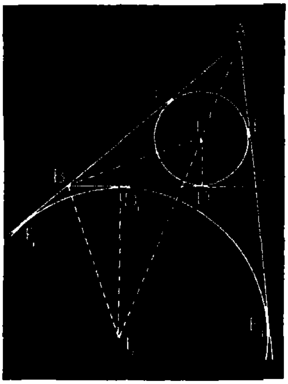


FIG. 53.

Shew that

$$\angle BI_1D_1 = \angle IBD = \frac{B}{2},$$

and hence

$$\frac{BD_1}{ID_1} = \tan\frac{B}{2} = \frac{ID}{BD},$$

$$\therefore\ r \cdot r_1 = (s-b)(s-c).$$

Similarly

$$r \cdot r_2 = (s-c)(s-a),$$

$$r \cdot r_3 = (s-a)(s-b).$$

74. Let ABC be a triangle, I, I_1 the centres of the inscribed circle and the escribed circle which touches BC and the other two sides produced.

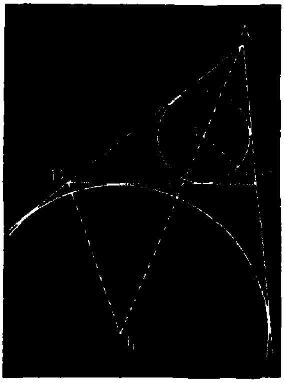


FIG. 54.

Shew that $\triangle$s AIC, ABI_1 are equiangular and hence, making use of formula (ii), that

$$\frac{AI}{AC} = \frac{AB}{AI_1}.$$

$$\therefore\ AI \cdot AI_1 = b \cdot c.$$

Similarly shew that

$$BI \cdot BI_2 = c \cdot a,$$

$$CI \cdot CI_3 = a \cdot b.$$

75. Let ABC be a triangle, I the centre of the inscribed circle DEF, r the radius of this circle.

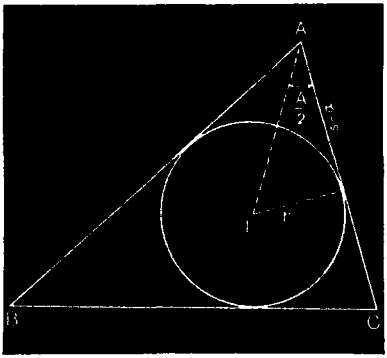


FIG. 55

Shew that AI bisects the angle BAC, and hence

$$\tan\frac{A}{2}=\frac{r}{s-a}$$

$$=\frac{\Delta}{s(s-a)} \text{ ....from Expt. 71.}$$

$$=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \text{ ......(see p. 102, v.)}$$

Similarly

$$\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}},$$

$$\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \text{ ....(vi).}$$

These formulae are adapted to logarithmic computation, and are those generally used for the solution of triangles of which *the three sides* are given.

Formulae (v) and (vi), together with corresponding expressions for the sines and cosines of the half-angles can be obtained from Expts. **73, 74**.

76. Let ABC be a triangle, I, I_1 the centres of the inscribed circle DEF and the escribed circle $D_1E_1F_1$ which touches BC and the other two sides produced. Join IE, I_1E_1.

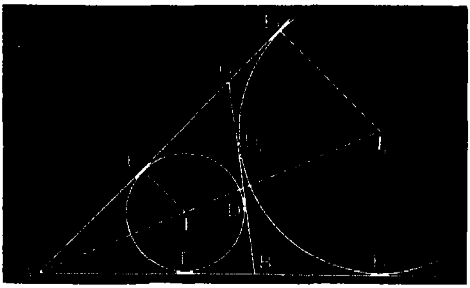


FIG. 56.

Then $\Delta^2 = r \cdot r_1 \cdot s \cdot (s-a)$ ....Expts. **71, 72**,

$= s(s-a)(s-b)(s-c)$,

$$\sin^2\frac{A}{2} = \frac{r}{AI} \cdot \frac{r_1}{AI_1}$$

$$= \frac{(s-b)(s-c)}{bc} \text{ ......Expts. 73, 74,}$$

$$\cos^2\frac{A}{2} = \frac{AE}{AI} \cdot \frac{AE_1}{AI_1}$$

$$= \frac{s(s-a)}{bc} \text{ ....Expt. 74,}$$

$$\tan^2\frac{A}{2} = \frac{r}{AE} \cdot \frac{r_1}{AE_1}$$

$$= \frac{(s-b)(s-c)}{s(s-a)} \text{ ....Expt. 73.}$$

The "Sine Rule" can be used with logarithms to solve a triangle in which (i) *one side and two angles*, and (ii) *two sides and an opposite angle* are given: the following experiment will give us a formula for the remaining case in which *two sides and the included angle are given.*

77. Let ABC be a triangle in which $\angle A > \angle B$: with centre C and radius CA, describe a circle cutting BC in D and BC produced in E; join AD, AE and draw DF parallel to EA to cut BA in F.

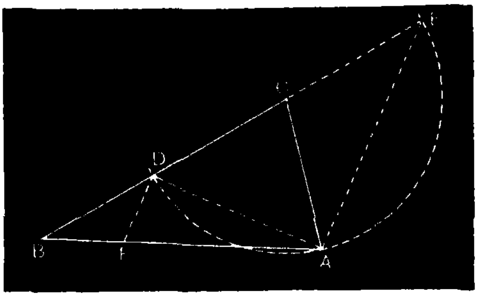


FIG. 57.

Then $\angle FDA = \angle DAE =$ a rt. $\angle$.

Also $\angle EDA = \frac{1}{2}\angle ECA = \frac{1}{2}(A+B)$,

$\therefore \angle DAF = \angle EDA - \angle DBA = \frac{1}{2}(A-B)$;

and $BD = a - b$, $BE = a + b$.

Hence

$$\frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{DF}{DA} \div \frac{AE}{DA} = \frac{DF}{AE}.$$

But, since BDF, BEA are equiangular Δs,

$$\therefore \frac{DF}{AE} = \frac{BD}{BE} = \frac{a-b}{a+b}.$$

$$\therefore \frac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)} = \frac{a-b}{a+b},$$

$$\text{i.e., } \tan\tfrac{1}{2}(A-B) = \frac{a-b}{a+b}\cot\tfrac{1}{2}C.$$

Exercises.

1. Obtain the following expression for R, the radius of the circle circumscribed to the triangle ABC :—

(i) $R = \dfrac{a}{2 \sin A}$, (ii) $R = \dfrac{abc}{4\Delta}$.

2. Verify the formulae

$$r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2},$$

$$r_1 = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

3. Shew that

$$r_1 + r_2 + r_3 - r = 4R,$$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}.$$

4. The sides of a right-angled triangle are 12, 16, 20: find r, r_1, r_2, r_3, R.

5. The sides of a triangle are

(i) 17, 25, 26; (ii) 11, 13, 20;

(iii) 35, 44, 75; (iv) 13, 20, 21.

Find, in each case, the angles of the triangle and its area.

6. Two angles of a triangle are 59°, 61°; the side adjacent to both is 10 inches long; find the difference in area between this triangle and an equilateral triangle on the same base.

7. The difference between two angles of a triangle is a right angle: the two sides opposite these angles are in the ratio of 2 : 1: find the angles of the triangle.

8. By means of the formulae given in Expt. 76 verify that

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2},$$

$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$$

$$= 2 \cos^2 \frac{A}{2} - 1$$

$$= 1 - 2 \sin^2 \frac{A}{2}.$$

§ 18. Solution of Triangles and Problems.

By the use of seven-figure logarithms it can be shewn that the following are the six "parts" of a certain triangle **ABC**:—

$$a = 19828 \qquad \mathsf{A} = 28^\circ\ 33'\ 56{\cdot}8'',$$
$$b = 37624 \qquad \mathsf{B} = 65^\circ\ \ 8'\ 20{\cdot}8'',$$
$$c = 41380 \qquad \mathsf{C} = 86^\circ\ 17'\ 42{\cdot}4''.$$

This triangle is solved from different data, by the three standard formulae, by the five-figure tables given on pp. 88—95, in Ex. 1—5. These worked-out exercises will serve, not only to shew the degree of accuracy of five-figure logarithms, but also as model solutions.

When calculating the "possible error," it should be remembered that the last figure in five-figure tables is thus determined from seven-figure tables. If the sixth and seventh figures taken together are less than 50, they are omitted; whilst if they are over 50, the fifth figure is increased by 1.

For example, consider the following:—"Find log 329·34."

From seven-figure tables

$$\log 3{\cdot}2900 = 0{\cdot}5171959, \quad \log 3{\cdot}3000 = 0{\cdot}5185139,$$

$\therefore$ with five-figure tables

$$\log 3{\cdot}2900 = 0{\cdot}51720, \qquad \log 3{\cdot}3000 = 0{\cdot}51851,$$

and by interpolation

$$\log 3{\cdot}2934 = 0{\cdot}51765,$$
$$\therefore \quad \log 329{\cdot}34 = 2{\cdot}51765.$$

Now with seven-figure tables it is found that

$$\log 329{\cdot}34 = 2{\cdot}5176445$$
$$= 2{\cdot}51764 \text{ to five figures.}$$

Hence there is an error $= 1$ in the fifth figure.

Again, for example :—"Find log 1·1405."

From seven-figure tables,

$\log 1{\cdot}1400 = 0{\cdot}0569049, \quad \log 1{\cdot}1500 = 0{\cdot}0606978,$

$\therefore$ with five-figure tables,

$\log 1{\cdot}1400 = 0{\cdot}05690, \qquad \log 1{\cdot}1500 = 0{\cdot}06070,$

and by interpolation

$$\log 1{\cdot}1405 = 0{\cdot}05709.$$

Now with seven-figure tables it is found that

$$\begin{aligned}\log 1{\cdot}1405 &= 0{\cdot}0570953\\ &= 0{\cdot}05710 \text{ to five figures,}\end{aligned}$$

and there is again an error of 1 in the fifth figure.

Hence in the following exercises it may be taken that there is a *possible* error $= \pm 1$ in each logarithm taken, and supposing that the worst happens, i.e. that the error is of one sign for those logarithms which have to be added, and of the other sign for those which have to be subtracted in any particular problem, the greatest possible error is equal to the number of separate logarithms used in the calculation. Thus in using the formula

$$\log \tan^2 \frac{A}{2} = \log (s-b) + \log (s-c) - \log s - \log (s-a)$$

we *may* have an error $= \pm 4$ in the fifth figure; this would of course give a possible error $= \pm 2$ in the value of $L \tan \frac{A}{2}$.

NOTE. *It should be noticed that the difference between the five-figure* **logarithms obtained (i) by interpolation and (ii) by direct approximation from seven-figure tables has been taken as the** *greatest possible error*: **these values are however generally one on each side of the true value, differing only by ·5 in the fifth place from their true value, as in the examples given above.**

1. **Three sides.**

Given $a=19828$, $b=37624$, $c=41380$; *find by separate calculations the angles* A, B, C.

[*Model Solution.*]

Formulae used.

$$(1)\ \tan^2 \tfrac{1}{2} \mathrm{A}=\frac{(s-b)(s-c)}{s(s-a)},$$

$$(2)\ \tan^2 \tfrac{1}{2} \mathrm{B}=\frac{(s-c)(s-a)}{s(s-b)},$$

$$(3)\ \tan^2 \tfrac{1}{2} \mathrm{C}=\frac{(s-a)(s-b)}{s(s-c)}.$$

Data.

$$\begin{array}{rcr} a=19828 & \therefore & s=49416 \\ b=37624 & & s-a=29588 \\ c=41380 & & s-b=11792 \\ \hline 2s=98832\ . & & s-c=\ 8036. \end{array}$$

Hence (i) $\log \tan^2 \frac{1}{2} \mathrm{A}=\log 11792+\log 8036-\log 49416-\log 29588$

$$\begin{array}{rrr} = & 4{\cdot}06819 & -4{\cdot}69373 \\ & 340 & 14 \\ & 3{\cdot}90472 & 4{\cdot}46982 \\ & 32 & 129 \\ = & 7{\cdot}97663 & \\ & -9{\cdot}16498 & \\ = & \bar{2}{\cdot}81165 & \end{array}$$

$$\begin{array}{rrl} \therefore\ \ \mathrm{L}\tan \frac{1}{2} \mathrm{A}= & \bar{1}{\cdot}40583+10 & \\ = & 9{\cdot}40583 & \text{diff.} \qquad =371 \\ \mathrm{L}\tan 14^\circ\ 10' & 212 & \\ 20' & 742 & \text{diff. for } 10'=530, \end{array}$$

$$\therefore\ \ \tfrac{1}{2} \mathrm{A}=14^\circ\ 10'+\tfrac{371}{530}\times 10'$$
$$=14^\circ\ 17',$$
$$\therefore\ \ \mathrm{A}=\underline{28^\circ\ 34'}.$$

NOTE. Here the possible error in $\log \tan^2 \frac{1}{2}$A is ·00004, leading to possible errors of ·00002 in $L \tan \frac{1}{2}$A, of $2\frac{1}{2}''$ in $\frac{\mathrm{A}}{2}$ (since diff. for $10'=530$) and $5''$ in A. As the answer is only $3{\cdot}2''$ wrong, some of the errors above have either not existed or cancelled one another. (Cf. Note, p. 110.)

Again (ii) $\log \tan^2 \frac{B}{2} = \log 8036 + \log 29588 - \log 49416 - \log 11792$

$$
\begin{array}{rll}
= & 3{\cdot}90472 & -4{\cdot}69373 \\
 & 32 & 14 \\
 & 4{\cdot}46982 & 4{\cdot}06819 \\
 & 129 & 340 \\
\hline
= & 8{\cdot}37615 & \\
 & -8{\cdot}76546 & \\
\hline
= & \bar{1}{\cdot}61069 &
\end{array}
$$

$$
\begin{array}{lll}
\therefore\ \ \text{L} \tan \frac{B}{2} = 9{\cdot}80535 & \text{diff.} & = 116. \\
\text{L} \tan 32^\circ\, 30' \quad\ \ 419 & & \\
\qquad\quad\ \ 40' \quad\ \ 697 & \text{diff. for } 10' = 278. &
\end{array}
$$

$$\therefore\ \ \frac{B}{2} = 32^\circ\ 34'\ 10'',$$

$$\therefore\ \ B = \underline{65^\circ\ 8'\ 20''}.$$

Similarly (iii) $\log \tan^2 \frac{C}{2} = \bar{1}{\cdot}94379$;

$$\therefore\ \ L \tan \frac{C}{2} = 9{\cdot}97190,$$

$$\therefore\ \ C = 86^\circ\ 17'\ 42''.$$

Hence we have

$$
\begin{array}{l}
A = 28^\circ\ 34' \\
B = 65^\circ\ \ 8'\ 20'' \\
C = 86^\circ\ 17'\ 42'' \\
\hline
\quad\ \underline{180^\circ\ \ 0'\ \ 2''}.
\end{array}
$$

2. Two sides and the included angle.

Given $b = 37624$, $c = 41380$, $A = 28^\circ\ 33'\ 56{\cdot}8''$; *find* B, C *and* a.

[*Model Solution.*]

Formulae used

(1) $\tan \frac{1}{2}(C - B) = \dfrac{c - b}{c + b} \tan \frac{1}{2}(C + B)$.

(2) $a = \dfrac{b \sin A}{\sin B}$.

Data.

$c = 41380$ $\qquad$ $A = 28° \; 33' \; 56{\cdot}8''$

$b = 37624$ $\qquad \therefore \quad C + B = 151° \; 26' \; 3{\cdot}2''$

$\therefore \quad c - b = 3756$ $\qquad \therefore \quad \frac{1}{2}(C+B) = 75° \; 43' \; 1{\cdot}6''$.

$c + b = 79004$.

Hence (i) $L \tan \frac{1}{2}(C - B) = \log 3756 + L \tan 75° \; 43' \; 1{\cdot}6'' - \log 79004$

$$
\begin{array}{rrl}
= & 3{\cdot}57403 & \quad -4{\cdot}89763 \\
 & 70 & \qquad\qquad 2 \\
 & 10{\cdot}59258 & \\
 & 160 & \\
= & 14{\cdot}16891 & \\
- & 4{\cdot}89765 & \leftarrow \\
= & 9{\cdot}27126 & \text{diff.} \quad = 329. \\
L \tan 10° \; 30' & 6797 & \\
40' & 7496 & \text{diff. for } 10' = 699.
\end{array}
$$

$\therefore \quad \frac{1}{2}(C - B) = 10° \; 30' + \frac{329}{699} \times 10'$

$= 10° \; 34' \; 42{\cdot}3''$.

But $\qquad \frac{1}{2}(C + B) = 75° \; 43' \; 1{\cdot}6''$,

$\therefore \quad C = \underline{86° \; 17' \; 43{\cdot}9''}$

$B = \underline{65° \; 8' \; 19{\cdot}3''}$.

Again (ii) $\log a = \log 37624 + L \sin 28° \; 33' \; 56{\cdot}8'' - L \sin 65° \; 8' \; 19{\cdot}3''$.

$$
\begin{array}{rrl}
= & 4{\cdot}57519 & \quad -9{\cdot}95728 \\
 & 28 & \qquad\qquad 48 \\
 & 9{\cdot}67866 & \\
 & 91 & \\
= & 14{\cdot}25504 & \\
- & 9{\cdot}95776 & \leftarrow \\
= & 4{\cdot}29728 & \text{diff.} \quad = 61. \\
\log 1{\cdot}98 & 667 & \\
9 & 885 & \text{diff. for } 1 = 218.
\end{array}
$$

$\therefore \quad a = 1{\cdot}98\tfrac{61}{218} \times 10^4$

$= \underline{19828.}$

NOTE. *In the last example* the possible error in $L\tan\frac{1}{2}(\mathsf{C}-\mathsf{B})$ is ·00008: this, in that part of the table corresponding to the approximate value of $\frac{1}{2}(\mathsf{C}-\mathsf{B})$, where the difference for $10'$ is 699, corresponds to a possible error of less than $3''$ in $\frac{1}{2}(\mathsf{C}-\mathsf{B})$, and hence in C and in B. This error might affect the finding of a, but the difference for $10'$ in L sines between $65°$ and $66°$ is less than 60, and hence an error of $3''$ will not affect the figure in the fifth decimal place.

3. Two sides and the angle opposite the smaller of these. (*The ambiguous case.*)

Given $a=19828$, $c=41380$, $\mathsf{A}=29°\,33'\,56{\cdot}8''$; *solve the triangle.*

[*Model Solution.*]

<u>Formula used.</u> $\dfrac{a}{\sin\mathsf{A}}=\dfrac{b}{\sin\mathsf{B}}=\dfrac{c}{\sin\mathsf{C}}$.

<u>Data.</u> $a=19828$ $\quad\mathsf{A}=28°\,33'\,56{\cdot}8''$.
$c=41380$.

Hence $L\sin\mathsf{C}=\log 41380-\log 19828+L\sin 28°\,33'\,56{\cdot}8''$

$$
\begin{array}{rrl}
 = & 4{\cdot}61595 & \quad -4{\cdot}29667 \\
 & 84 & \qquad\quad 61 \\
 & 9{\cdot}67866 & \\
 & 91 & \\
 = & 14{\cdot}29636 & \\
 - & 4{\cdot}29728 & \leftarrow \\
 & 9{\cdot}99908 & \text{diff.} \quad =5 \\
L\sin 86°\,10'= & 3 & \\
20'= & 11 & \text{diff. for } 10'=8.
\end{array}
$$

$\because\ \mathsf{C}$ or $180°-\mathsf{C}=86°\,10'+\frac{5}{8}\times 10'$
$=86°\,16'\,15''$

$\therefore\ \mathsf{C}=\underline{86°\,16'\,15''}$ or $\underline{93°\,43'\,45''}$.

Case I. If

$\mathsf{C}=86°\,16'\,15''$,

Then, $\because\ \mathsf{A}=28°\,33'\,57''$ (approx.)

$\therefore\ \mathsf{B}=65°\ \ 9'\,48''$

$\overline{180°\ \ 0'\ \ 0''}$

Hence $\log b = \log 19828 + L\sin 65^\circ\, 9'\, 48'' - L\sin 28^\circ\, 33'\, 57''$.

$$
\begin{array}{lrrl}
 & = & 4{\cdot}29667 & \quad -9{\cdot}67866 \\
 & & 61 & \quad \phantom{-9{\cdot}678}91 \\
 & & 9{\cdot}95728 & \\
 & & 55 & \\
 & = & 14{\cdot}25511 & \\
 & - & 9{\cdot}67957 & \\
 & = & 4{\cdot}57554 & \text{diff.} \quad = 35, \\
\log 3\,76 & & 519 & \\
7 & & 634 & \text{diff. for } 1 = 115,
\end{array}
$$

$\therefore\ b = 3{\cdot}76\tfrac{35}{115} \times 10^4$

$= \underline{37630.}$

Case II. If $\mathsf{C} = 93^\circ\, 43'\, 45''$

Then, $\because\ \mathsf{A} = 28^\circ\, 33'\, 57''$ (approx.)

$\therefore\ \mathsf{B} = \underline{57^\circ\, 42'\, 18''}$

$180^\circ\ 0'\ 0''$

$\therefore\ \log c = \log 19828 + L\sin 57^\circ\, 42'\, 45'' - L\sin 28^\circ\, 33'\, 57''$.

$$
\begin{array}{lrrl}
 & = & 4{\cdot}29667 & \quad -9{\cdot}67866 \\
 & & 61 & \quad \phantom{-9{\cdot}678}91 \\
 & & 9{\cdot}92683 & \\
 & & 22 & \\
 & = & 14{\cdot}22433 & \\
 & - & 9\,67957 & \\
 & = & 4{\cdot}54476 & \text{diff.} \quad = 69, \\
\log 3{\cdot}50 & & 407 & \\
1 & & 531 & \text{diff. for } 1 = 124,
\end{array}
$$

$\therefore\ c = 3{\cdot}50\tfrac{69}{124} \times 10^4$

$= \underline{35056.}$

NOTE. The possible error of ·00003 in L sin C corresponds to $\frac{3}{8}$ of 10′, i.e. about 4′ in C. In general, the results obtained from L sines when the angles are near 90° are very untrustworthy. This affects the values of b, and in such cases the more laborious method, of using the formula $a^2 = b^2 + c^2 - 2bc\cos \mathsf{A}$ to find b first, should be adopted. C can then be found from

$$\tan^2 \tfrac{1}{2}\mathsf{C} = \frac{(s-a)(s-b)}{s(s-a)}.$$

4. Two sides and the angle opposite the greater of these.

Given $a = 19828$, $b = 37624$, $\mathrm{B} = 65^\circ\ 8'\ 20{\cdot}8''$; *find* A.

[*Model Solution.*]

<u>Formula used.</u> $\quad \sin \mathrm{A} = \dfrac{a}{b} \sin \mathrm{B}.$

<u>Data.</u> $\quad a = 19828 \qquad \mathrm{B} = 65^\circ\ 8'\ 20{\cdot}8''.$

$\qquad\qquad b = 37624.$

Hence $\quad L \sin \mathrm{A} = \log 19828 - \log 37624 + L \sin 65^\circ\ 8'\ 20{\cdot}8''$

$$
\begin{array}{rrll}
 & = & 4{\cdot}29667 & \quad -4{\cdot}57519 \\
 & & 61 & \qquad\quad\ \ 28 \\
 & & 9{\cdot}95728 & \\
 & & 48 & \\
 & = & 14{\cdot}25504 & \\
 & - & 4{\cdot}57547 & \\
 & = & 9{\cdot}67957 & \text{diff.} \qquad = 91. \\
L \sin 28^\circ\ 30' & = & 7866 & \\
40' & & 8098 & \text{diff. for } 10' = 232, \\
\therefore\ \mathrm{A} & = & 28^\circ\ 30' + \frac{91}{232} \times 10' & \\
 & = & \underline{28^\circ\ 33'\ 55''}\ * . &
\end{array}
$$

NOTE. Here the possible error in $L \sin \mathrm{A}$ may be ·00003 which corresponds to a possible error of over 6″ in A.

* The supplementary value 151° 26′ 1″ is inadmissible for A + B would then be equal to 216° 34′ 21·8″, which is impossible.

5. One side and two angles.

Given $a = 19828$, $\mathrm{B} = 65^\circ\ 8'\ 20{\cdot}8''$, $\mathrm{C} = 86^\circ\ 17'\ 42{\cdot}4''$; *solve the triangle.*

[*Model Solution.*]

<u>Formula used.</u> $\quad \dfrac{a}{\sin \mathrm{A}} = \dfrac{b}{\sin \mathrm{B}} = \dfrac{c}{\sin \mathrm{C}}.$

<u>Data.</u> $\quad a = 19828. \qquad \mathrm{B} = 65^\circ\ 8'\ 20{\cdot}8''$

$\qquad\qquad\qquad\qquad\qquad \mathrm{C} = 86^\circ\ 17'\ 42{\cdot}4''$

$\qquad\qquad\qquad\qquad\quad \therefore\ \mathrm{A} = 28^\circ\ 33'\ 56{\cdot}8''.$

Hence (i) $\log b = \log 19828 + L\sin 65^\circ\, 8'\, 20{\cdot}8'' - L\sin 28^\circ\, 33'\, 56{\cdot}8''$.

$$
\begin{array}{lrl}
 & = \quad 4{\cdot}29667 & \qquad -9{\cdot}67866 \\
 & 61 & \qquad\qquad\quad 91 \\
 & 9{\cdot}95728 & \\
 & 48 & \\
 & = \quad 14{\cdot}25504 & \\
 & - \quad 9{\cdot}67957 & \\
 & = \quad 4{\cdot}57547 & \text{diff.} \qquad = 28, \\
\log 3{\cdot}76 & 519 & \\
7 & 634 & \text{diff. for } 1 = 115,
\end{array}
$$

$\therefore \quad b = 3{\cdot}76\tfrac{28}{115} \times 10^4$

$= \underline{37624.}$

Again (ii) $\log \mathrm{C} = \log 19828 + L\sin 86^\circ\, 17'\, 42{\cdot}4'' - L\sin 28^\circ\, 33'\, 56{\cdot}8''$

$$
\begin{array}{lrl}
 & = \quad 4{\cdot}29667 & \qquad -9{\cdot}67866 \\
 & 61 & \qquad\qquad\quad 91 \\
 & 9{\cdot}99903 & \\
 & 6 & \\
 & - \quad 14{\cdot}29637 & \\
 & - \quad 9{\cdot}67957 & \\
 & = \quad 4{\cdot}61680 & \text{diff.} \qquad = 85. \\
\log 4{\cdot}18 & 595 & \\
4 & 700 & \text{diff. for } 1 = 105,
\end{array}
$$

$\therefore \quad c = 4{\cdot}13\tfrac{85}{105} \times 10^4$

$= \underline{41381.}$

6. Find the area of the triangle whose sides are 384·11 ft., 564·67 ft., 663·44 ft., in acres, roods and poles to the nearest pole.

7. The sides of a triangle are 13, 9, 12; find the greatest angle of the triangle.

8. Given $b = 105$, $c = 55$, $\mathrm{A} = 51^\circ$, find B and C.

9. Given $a = 1770{\cdot}1$, $b = 2164{\cdot}5$, $\mathrm{A} = 35^\circ\, 36'\, 20''$; find B, C and c.

10. Find all the parts of the triangles which have one side 90 ft. long, another side 60 ft. long, and the angle opposite to the shorter side equal to 18° 37′.

11. Two sides of a triangle are 200 ft. and 300 ft. respectively, the area is 20,000 square feet, calculate the angle between the two sides. Explain the two results that are obtained. Calculate the remaining side in each case.

12. From two stations A and B on shore, 3,742 yds. apart, a ship C is observed out at sea. The angles BAC, ABC are simultaneously observed to be 72° 34′ and 81° 41′ respectively. Find the distance of the ship from A.

13. From the top of a vertical cliff 100 ft. high, forming one bank of a river, the angles of depression of the top and bottom of a vertical cliff forming the opposite bank are 28° 40′ and 64° 30′ respectively. Find the height of the cliff on the opposite bank, and the breadth of the river.

14. P is a point vertically over N, a point in a horizontal plane, A and B are two points in the plane; AN is 100 ft., and the angles of elevation of P at A and B are 24° 30′ and 8° 10′ respectively: find the distance BN. [First find PN.]

15. AB is a line 250 ft. long, in the same horizontal plane as the foot, D, of a tower CD; the angles DAB and DBA are respectively 61° 23′ and 47° 14′; the angle of elevation of C from A is 34° 50′; find the height of the tower.

16. A and B are two places on opposite sides of a mountain; C is a distant landmark visible from A and B, AC = 10 miles, BC = 8 miles, the elevations of A and B at C are 8° and 2° 48′ respectively, and the angle BCA is 63° 28′. A tunnel has to be bored from A to B. Calculate

(1) the difference in height between A and B;

(2) the angle it makes with the horizon;

(3) the angle it makes with the direction AC;

(4) its length to nearest yard.

§ 19. Surveying Instruments.

As has been pointed out in § 2, a protractor whose radius is four inches, graduated in degrees, has the divisions one-fifteenth of an inch long: these could easily be divided into fifths, or odd fifths might be estimated with fair accuracy by eye. Errors of drawing, and even the breadth of the pen or pencil lines used, make it useless to strive for any greater degree of accuracy when solving problems graphically.

Now since in surveying and astronomical work generally, problems are solved by calculations with tables—which can be calculated to any degree of accuracy—accuracy of solution is only limited by the degree of accuracy with which measurements can be made.

By means of machinery, the protractor referred to above could be graduated into minutes or even seconds of arc; but the lines would have to be so excessively fine and close together, that the scale would be practically useless, even with a magnifying glass or low power "reading microscope." This excessive subdivision of a straight or circular scale is obviated by an auxiliary scale, sliding in contact with the main scale; this auxiliary scale is called a **vernier** after its inventor.

Verniers are of two kinds, "forward-reading" and "backward-reading." The principle is the same for both.

The value of the vernier in improving accuracy of measurements depends on the fact that the eye detects with considerable accuracy whether two straight lines are in exact alignment or not.

The "Forward-reading Vernier."

The simplest form of vernier, and the usual one for straight scales, is the *decimal* vernier.

Let **AB** be a scale of inches and tenths, let **V** be an auxiliary scale, $\frac{9}{10}$ in. long, divided into ten equal parts: then the difference between a scale division and a vernier division is

$$\tfrac{1}{10} - \tfrac{1}{10} \cdot \tfrac{9}{10} = \tfrac{1}{100} \text{ in.}$$

Hence, if any vernier division is opposite to a scale division, moving the vernier $\frac{1}{100}$ in. to the right will bring the next vernier division opposite a scale division. Hence, if the vernier divisions are numbered consecutively from left to right, the number attached to a division on the vernier, which comes opposite to (or most nearly so) a scale division, gives the number of hundredths of an inch which the vernier has been moved from the position in which its left-hand edge came opposite a scale division. Thus, in Fig. 58, the length of the pencil is equal to 3·6 inches + the amount the vernier has been moved to the right from the position in which the left-hand edge was opposite "3·6" on the scale. Since the division marked 4 on the vernier is opposite a scale division, it is seen that the vernier has been moved to the right $4 \times \cdot 01$ in. Hence the length of the piece of pencil is

3·64 inches.

It will readily be seen that the vernier zero need not be at the edge of the auxiliary scale, so long as it is distant from it an exact number of tenths of an inch; and similarly for the other end of the vernier.

1. Read the height of the barometer, in Fig. 59, (i) to ·002 in. (right-hand side), (ii) to ·1 mm. (left-hand side).

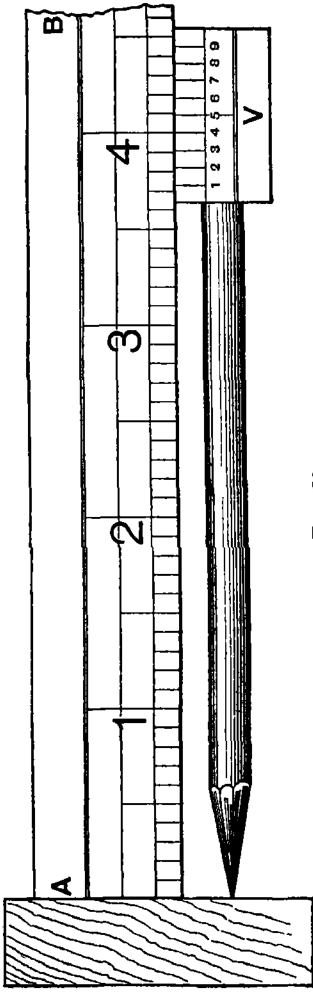

B
4
3
2
1
A
1 2 3 4 5 6 7 8 9
V

Fig. 58.

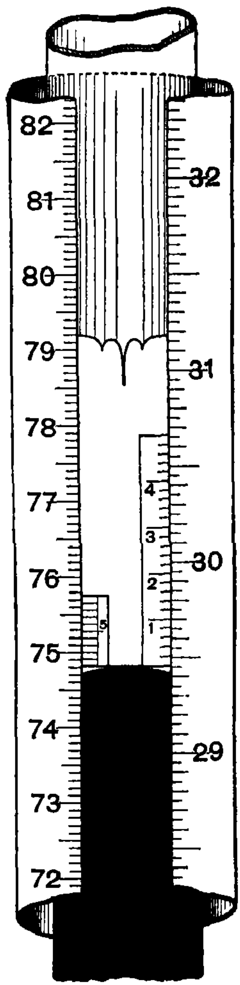

82
81
80
79
78
77
76
75
74
73
72
32
31
30
29
4
3
2
1
5

Fig. 59.

Verniers, on the same principle as the decimal vernier, may be constructed, in which any arbitrary number (n) of divisions may be taken from the scale and the total length of these n divisions subdivided on the vernier into $(n+1)$ equal parts, the vernier so constructed reading to $\dfrac{1}{n+1}$th of a scale subdivision.

For instance, with a scale in inches and tenths, take 19 tenths on a vernier divided into 20 equal parts: then difference between a scale division and a vernier division is

$$\tfrac{1}{10} - \tfrac{1}{20} \cdot \tfrac{19}{10} = \tfrac{1}{200},$$

and the vernier reads to ·005 inch, *although the distance between the graduations on the vernier is very little less than one-tenth of an inch.*

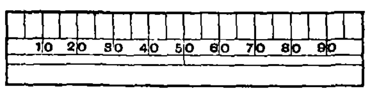


FIG. 60.

Fig. 60 shews how this vernier should be numbered; the division 85 on the vernier, for instance, if coincident with a subdivision on the scale, standing for an addition of ·085 in., to the inches and tenths, observed directly from the scale, in the length to be measured.

With such a vernier a reading microscope would be necessary; for with the naked eye, there would be great difficulty in detecting which, out of a group of several consecutive lines on the vernier, was the one in most accurate alignment with a division on the scale.

2. Construct a vernier for use with a scale of inches and eighths to read to a thirty-second of an inch [here $n=3$].

Although *theoretically* a scale with a sufficiently long vernier can thus be constructed to read to any degree of accuracy, *practically* there is a limit imposed owing to the fact that the number of vernier lines which are in approximate alignment with corresponding scale divisions becomes greater as the difference between the subdivisions on the vernier and scale gets less.

The "Backward-reading Vernier."

If n scale divisions are taken and divided into $n-1$ equal parts; then each division on the vernier is to the right of the corresponding scale division by $\frac{1}{n-1}$ of a scale division, if we count as before from the left-hand end of the vernier scale; if however we count from the right-hand end, the vernier divisions are to the left of the corresponding scale divisions as before.

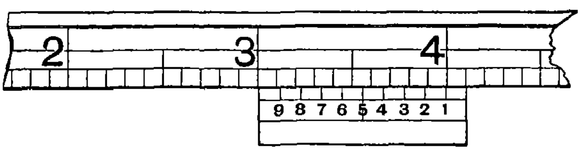


FIG. 61.

Thus, Fig. 61 shews a scale in inches and tenths with a vernier, on which eleven-tenths of an inch have been taken and divided into 10 equal parts, and numbered as shown. This scale and vernier read to ·01 in.

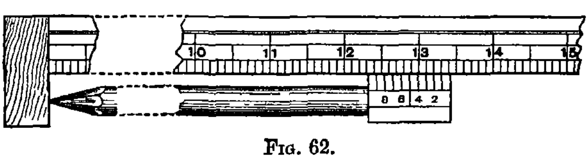


FIG. 62.

In Fig. 62 the length of the pencil is 12·32 cm.

*General theory of the vernier.

Let AB be part of a scale graduated in units and sub-units; let there be m sub-units in a whole unit.

Let CD, the vernier, be another scale capable of sliding with its edge always in contact with the edge of the scale AB.

Mark off on the vernier a distance equal to n sub-units of the scale, and subdivide this distance into $p\,(>n)$ equal parts.

Then the distance between any two marks on the vernier is equal to $\frac{1}{p}\cdot\frac{n}{m}$ units of scale.

$\therefore$ difference between the subdivisions on the scale and the vernier is equal to

$$\frac{1}{m}-\frac{1}{p}\cdot\frac{n}{m}=\frac{p-n}{p\,.\,m}\text{ units.}$$

Although p can be any number greater than n, for convenience it is almost always taken equal to $n+1$, thus making the difference between the subdivisions equal to $\frac{1}{pm}$ units.

In consequence, the rth division of the vernier is $\frac{r}{pm}$ units to the left of the corresponding scale division; and, since $\frac{r}{pm}$ is never greater than $\frac{1}{m}$, there is no ambiguity caused by the rth vernier division falling past the $(r-1)$th division of the scale.

The importance of this may be more readily seen by considering the following example.

A scale is divided into units and fifths: on the vernier a length equal to three-fifths of a unit is divided into eight parts.

Each division on the vernier is therefore equal to $\frac{3}{40}$ths of a unit. Hence the first vernier division falls to the left of the nearest scale division by $\frac{1}{5}-\frac{3}{40}=\frac{5}{40}$ unit;

the second vernier division,			by $\frac{1}{5}-\frac{6}{40}=\frac{2}{40}$ unit;
the third	,,	,,	by $\frac{2}{5}-\frac{9}{40}=\frac{7}{40}$ unit;
the fourth	,,	,,	by $\frac{2}{5}-\frac{12}{40}=\frac{4}{40}$ unit;
the fifth	,,	,,	by $\frac{2}{5}-\frac{15}{40}=\frac{1}{40}$ unit;
the sixth	,,	,,	by $\frac{3}{5}-\frac{18}{40}=\frac{6}{40}$ unit;
the seventh	,,	,,	by $\frac{3}{5}-\frac{21}{40}=\frac{3}{40}$ unit.

* These two pages may be omitted on first reading.

Hence the vernier must be numbered as in Fig. 63: and, if any division of the vernier coincides with a division of the scale, it shews that the vernier has been moved a distance

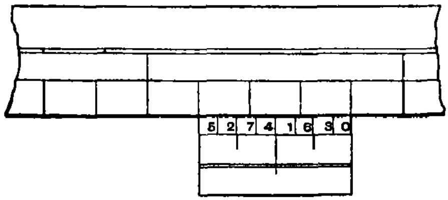


FIG. 63.

equal to r-fortieths of a unit (where r is the number attached to the vernier division in coincidence) to the right from the position in which the vernier zero coincided with a scale division.

If however $p=n+1$, the numbering is consecutive from the zero from left to right, as in the decimal vernier.

Suppose we have to measure a rod between 4 and 5 units long, and that when one end is placed exactly opposite the zero of the scale, the other end comes between the sixth and seventh subdivisions on the scale; also that, when the vernier (having previously been pushed along out of the way) is brought back with its zero end into contact with the rod, it is found that the rth division on the vernier coincides with a scale division. Then the length of the rod is 4 units+6 subdivisions+the distance the vernier zero has been moved from a position of coincidence with the sixth scale subdivision. Now in this latter position the distance of the rth vernier division from the nearest scale division to the right of it is $\frac{r}{pm}$ units; hence if these division marks now coincide the vernier must have been moved a distance to the right equal to $\frac{r}{pm}$ units.

Hence the length of the rod is

$$4+\frac{6}{m}+\frac{r}{pm} \text{ units.}$$

The principle is the same for the measurement of angles, the scale and vernier being annular and concentric: if a degree is divided into m equal parts on the scale, and if a length equal to n of these on the vernier are divided into $p\,(=n+1)$ equal parts, the scale and vernier will read to $\frac{1}{pm}$ of a degree.

Thus a 5-inch circle might be graduated at intervals of $5'$, the intervals being about $\frac{1}{100}$ inch long: on the vernier, about $1\frac{1}{2}$ inch long, a length equal to 149 of these intervals might be taken and divided into 150 equal parts: the scale and vernier would read to

$$\tfrac{1}{150}\cdot\tfrac{5}{60} \text{ of a degree} = 2'' \text{ of angle.}$$

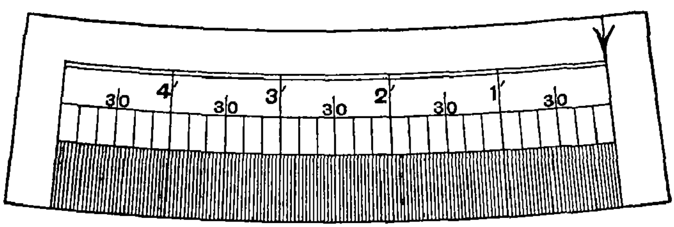


FIG. 64.

A reading microscope would of course be necessary: the vernier should be figured as in Fig. 64, which is enlarged to about twice natural size.

3. How would you divide a scale and vernier on a 4-inch circle to read to half-minutes, so that the vernier is not more than an inch long?

4. What is the radius of the smallest circle, which, with a vernier not more than 2 inches long, and the divisions on both scale and vernier not less than ·01 inch apart, will read to seconds?

The Surveyor's Level.

This instrument consists of a telescope mounted on a stand; it is fitted with levelling screws and two ordinary pattern "levels," one of which is, when accurately adjusted, parallel to the axis of the telescope and the other at right angles to it.

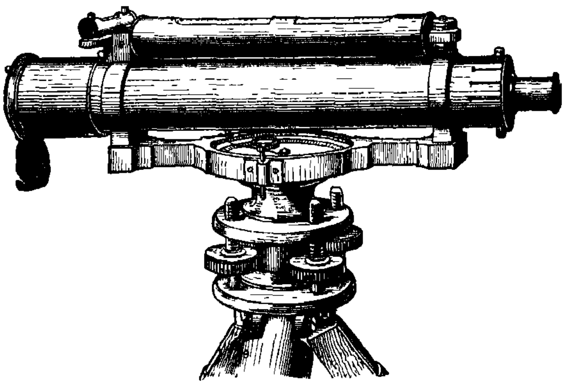


Fig. 65.*

A "compass" is fixed centrally under the telescope.

The Theodolite.

The simple theory of this instrument has been given in the introductory section. The explanation there given applies to the apparently complicated instrument, the '*Transit*' Theodolite (Fig. 67), the only essential difference being that the graduated vertical circle is rigidly fixed to the telescope (which corresponds with the pointer of Fig. 15), and turns with it, the rotation being measured against a fixed mark attached to the bearings of the telescope.

* From an instrument supplied by Messrs Griffin, London.

No verbal description, however well illustrated, can in any way take the place of a practical examination of such instruments as a theodolite, but it may enable the student to know what to look for, and where to find it, when he has the instruments in his hands; even though it is different in some small detail, such as disposition of the compass, different styles of locking-plates, by which the instrument is affixed to the stand, or different kinds of levelling-screw arrangement. A careful comparison of the figures of the theodolites on pp. 19, 131, and the level on p. 127 will be of service. A particularly handy arrangement for levelling the parallel plates, the upper of which carries the mounting of the telescope, and for centering the instrument above a given peg in the ground is shewn on the instrument in Fig. 67. This is a variation of the "Hoffman Tripod Head;" of which Fig. 66 shews an elevation and a section through the axis, giving all the essential features. It will readily be seen that the arrangement gives great freedom of adjustment. When the four milled levelling screws (of which two are shewn at **AA**) are all loosened, the upper part of the instrument can move round the centre of the lower ball **X** as a centre, in any direction, and also the centre of this lower ball and its socket and plate attached can be moved laterally in any direction. Thus, the instrument can be rapidly centred over a given mark, and roughly levelled. The milled screws are then turned, forcing the upper ball into its socket and locking the whole combination by friction.

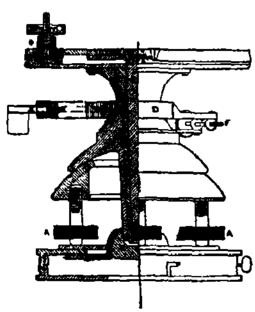


Fig. 66.

If now one pair of opposite screws be taken, releasing one of the pair and tightening the other, cant the whole of the top, until the axis of the ball and socket combination lies in the vertical plane containing the axes of the two screws used. Similar adjustment with the other pair of screws will then bring the axis exactly vertical. This is shewn by the pair of levels at right angles described below, one level being temporarily set parallel to each of the vertical planes containing the axes of a pair of levelling screws.

The axis of the ball and socket combination, now exactly vertical, is also the common axis of the two cones (B, C); to which the parallel plates are rigidly attached, at right angles to the common axis, the inner cone bearing the upper plate. Reference to Fig. 66 will shew that both plates can be freely rotated in a horizontal plane. The lower plate is rotated roughly into any required position, clamped to the ball and socket combination by means of the collar (DE), the tangent screw (F) providing for fine adjustment. The inner cone and upper parallel plate still have perfect freedom of rotation round the vertical axis; and after the upper plate has been placed roughly in any desired position, it is clamped firmly to the lower plate by means of the clutch (G), shewn in sectional elevation in Fig. 66. Fine adjustment is obtained by means of a tangent screw working against this clutch, the details being given (N) in Fig. 67 *a*. The lower plate is graduated, part of the scale being shewn at S in Fig. 66, and the upper plate bears a vernier V. Another vernier is generally added, diametrically opposite, to obviate errors of centring. Both are provided with reading microscopes.

The upper parallel plate (Fig. 67) carries a level (H), the fine adjustment for the horizontal circle, and the pillars

upon which the telescope is mounted. One of these pillars carries another level (K) accurately at right angles to the first, both levels being permanently adjusted parallel to the upper plate.

The telescope is mounted on a stout axle, at right angles to its "line of sight," this axle also carries the graduated vertical circle (LL) rigidly attached to it; the verniers (JJ) are attached to the arms of a T-piece, which can rotate freely round the axle of the telescope; its tail (W) being clamped to a cheek (X) on one of the pillars, and finely adjusted, by means of two screws (YY) working against one another; this arrangement is shewn in detail in Fig. 67 *b*. These screws are manipulated until the verniers both indicate zero*, when the telescope is pointing truly level, as indicated by the long delicate level (MM) attached to it parallel to its axis or "line of sight." When mounted the telescope can be roughly rotated by hand until the object to be observed is in the field of view. The clamp and tangent screw (N, O) combinations are then used to rotate the telescope horizontally and vertically, until the object comes accurately on the cross-wires, fixed in the telescope tube, which indicate the exact centre of the field of view.

Certain adjustments are called "permanent": these are generally manipulated by capstan screws as being less liable to be disarranged. The bearing at the top of one of the pillars is movable, allowing that end of the telescope axle to be raised or lowered, so as to render the axle perfectly* horizontal; frequently also there is a lateral motion at one end, bringing the axis of the telescope into a certain adjustment with the case of the compass. Other permanent adjustments are those of the three levels and the "line of sight or collimation" of the telescope. These should be frequently tested and set right if necessary.

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* That is, if the circle and T-piece are exactly centred: if not the error is noted.

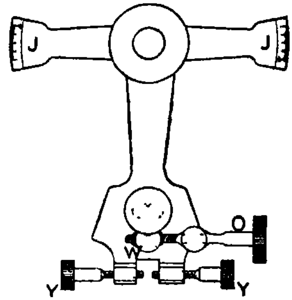
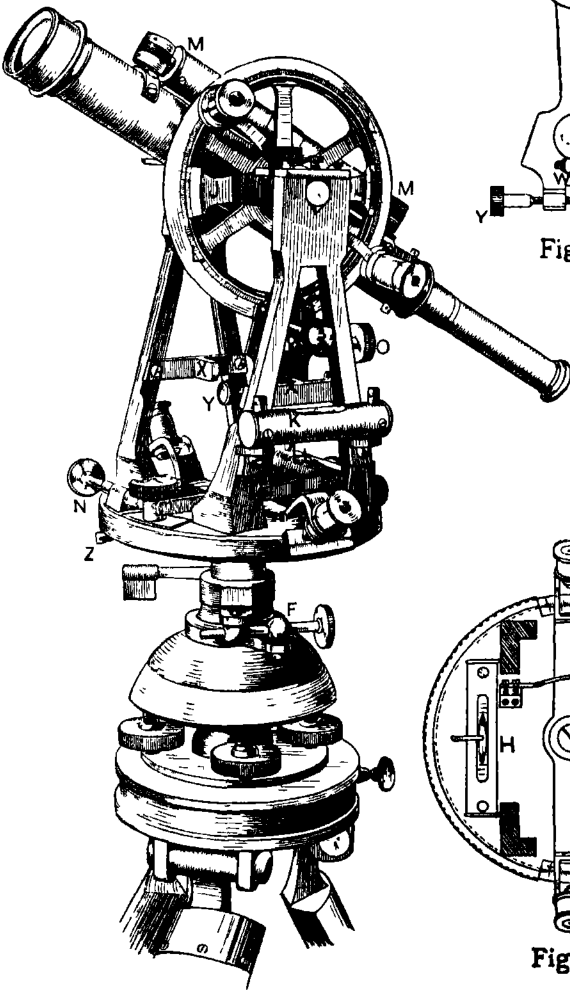


Fig.67(b)



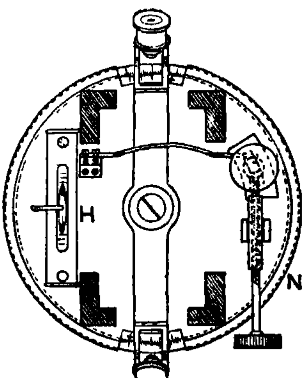


Fig.67(a)

FIG. 67.

To prepare for an observation.

(i) Set up the tripod, getting the top fairly level and approximately over the ground-mark.

(ii) Fix on the tripod head, by means of the bayonet catch and thumbscrew, or the locking plates, or other device.

(iii) Manipulate the levelling screws until the parallel plates are perfectly horizontal, and the axis of the instrument vertically over the peg, as indicated by the plumb-line attached, and clamp and adjust the lower plate.

(iv) Test the permanent adjustments for level by swinging the upper plate round into various positions and observing the readings of the levels.

To take an observation of the difference in bearing of two objects P and Q, and of the elevations or depressions of each.

(i) Move the telescope and upper plate until P appears in the field of view.

(ii) Clamp the horizontal and vertical circles, and by means of the tangent screws bring P exactly on the cross-wires.

(iii) Read the verniers for both circles and loose the clamps.

(iv) Repeat the operations for Q.

(v) Read the verniers on vertical circle when the level attached to the telescope shews that it is horizontal.

The difference in the readings of the vertical circle

in (iii) and (v) give the elevation or depression of P; of those in (iv) and (v) give the elevation or depression in Q; whilst the difference in the readings of the horizontal circle in (iii) and (iv) give the difference in bearing of P and Q. The compass in the instrument illustrated in Fig. 67*, being fixed in slots (Z) beneath the lower plate, may be so adjusted that the horizontal readings give the true bearing of each object observed.

* This illustration is from a photograph of an instrument supplied by Messrs J. Davis, Derby and London.

The Sextant.

The sextant is an instrument for measuring the angular altitude or elevation of an object or, in general, the angle subtended at the observer's eye by any two distant objects. Its use is almost entirely confined to observations at sea, for which an instrument like a theodolite, depending on a fixed horizon, would be unsuitable; whilst the sextant (by reason of the principle of its construction, which brings two *very distant* objects *simultaneously* into the field of view of the telescope) is independent of the rolling and pitching of the ship, its progressive motion, or of any fixed base.

Before the student can grasp the principle of the sextant, there are four experimental facts connected with Optics which he must understand.

I. A convex lens can be used for two things according to its position with regard to an object:—

(*a*) As a magnifying glass.

(*b*) To form a picture, called the **image**, of an object, such as is seen on the ground-glass screen in a camera.

By a suitable adjustment of the distance between them, two convex lenses can be so arranged that one, called the object-glass (O),

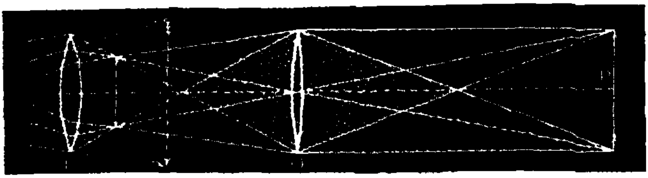


FIG. 68.

forms a reduced image Q of an object P, and the other, called the eye-piece (E), acts as a magnifying glass to observe this image. When the object is very distant, such as a star, the rays of light entering the object-glass are approximately parallel, and the image is formed at a point, at a fixed distance behind this lens, called the **focus.** It is at this point that the cross-wires of the telescope in a theodolite are fixed, so that they lie in the same plane as the image.

II. When half of the object-glass of a telescope is covered over, then an eye looking through the telescope will see exactly what it saw before except that the image is half as bright. This is readily seen from Fig. 69, where the rays proceeding from P, those entering the top half of the object-glass, are alone shewn.

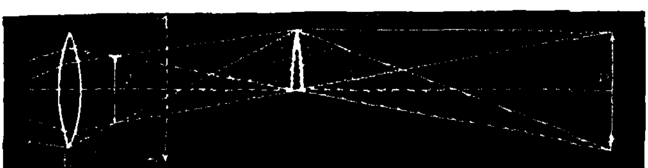


FIG. 69.

III. If a ray of light ABA′, travelling in a fixed direction in the plane of the paper, in Fig. 70, is incident on a plane mirror set up

perpendicular to the paper along PQ, and is reflected along the path BC, the angles ABP, CBQ are equal.

$$\therefore \angle A'BC = 2\angle A'BP.$$

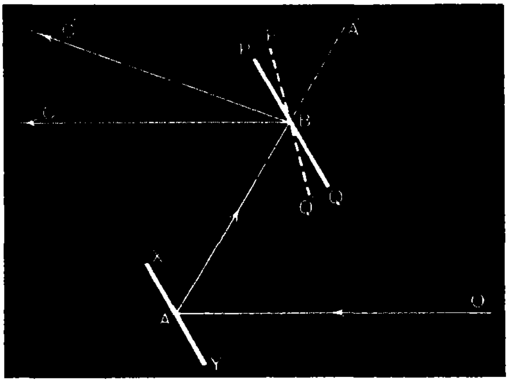


FIG. 70.

Again, if the mirror is rotated into the position denoted by P′Q′, the reflected ray will travel along the path BC′, such that

$$\angle A'BC' = 2\angle A'BP'.$$
$$\therefore \angle CBC' = 2\angle PBP'.$$

Also, if a **fixed** mirror XY had been used to reflect a ray, originally travelling in the fixed direction OA, along the fixed direction AB, it is evident that **when PQ is parallel to XY, BC is parallel to OA.** Hence the angle PBP′ is half the angle between BC′ and OA.

IV. The path of a ray of light is reversible. Hence, if a telescope is placed with its axis pointing along OA, and the mirror is rotated until the ray, from a distant object, travelling in the direction C′B, after being twice reflected, enters the telescope, then the angle through which the mirror has been rotated, from the position in which it was parallel to XY, is half the angle between BC and OA.

Suppose now that the mirror XY is replaced by a glass plate, of which only the half AY is silvered. If OA points towards an object S′, an image of S′ is formed in the telescope at *i*, whose brightness is half what it would be if the "mirror" XY was not there.

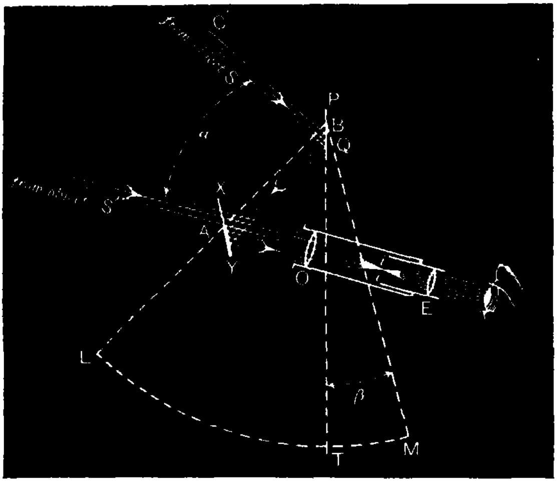


Fig. 71.

At the same time, an image of an object S, lying in the direction BC′, may be formed in the telescope at *i*, by reflection at the surfaces of PQ and the half mirror AY; these two images may be brought into contact in the case of the sun or star and the horizon, or into superposition in the case of two stars, since both are formed at the principal focus of the object-glass. The angle TBM will then measure half the angle subtended by the two objects at the observer's eye. This is shewn in Fig. 71.

The following is a picture of a Hadley's Sextant.

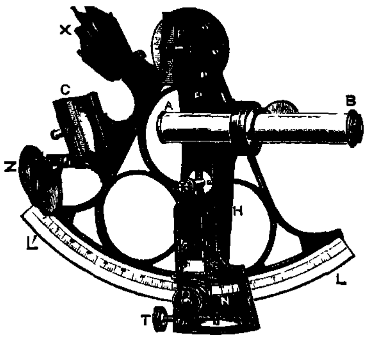


FIG. 72.*

AB is the telescope, **C** the fixed mirror corresponding to **XY** in Fig. 70, **F** the movable mirror corresponding to **PQ**.

PN is an arm by which the mirror **F** is turned; the amount of rotation being read off by means of the graduated arc **LL′**, the vernier **N**, and the reading microscope **M**; fine adjustment is provided for by the tangent screw, **T**. **H** is a handle to hold the instrument by, **X** and **Z** are blackened glasses to interpose between the light and the telescope when either of the objects viewed is too bright, in order to make the intensities approximately equal.

The graduations of the arc are figured with twice their actual values. Thus an actual arc of 5° would be figured as 10°; so that the reading of the arc is the actual angle subtended by the two objects.

* From an instrument supplied by Messrs Stanley, London.

TEST PAPERS.

§§ 1—5. 1.

1. Find the angle

   (*a*) between the directions E.N.E. by $\frac{1}{2}$E. and N.:

   (*b*) between the hands of a clock at 5.42.

2. A man is facing E.S.E.: he turns his back to a north-westerly wind which is blowing: through what angle does he turn?

3. A man starts from home for a walk at 10 a.m.; after walking for 3 miles he turns off to the right at an angle of 30°, and proceeds for 25 minutes: he then turns to the right again and walks along a road at right angles for 2 miles, when he comes to a road that takes him straight home. Assuming that the roads are straight, and that he maintains an even speed of $3\frac{3}{4}$ miles an hour, draw a map to scale of his walk and deduce the time he reaches home.

4. Two rods AB, BC, each 1 ft. long, stand on a table with B vertically over the middle point of AC: the ends A and C are joined by a piece of thin elastic also a foot long. If, under the influence of the weights of the rods, B sinks through a distance of $2\frac{1}{2}$ inches, find the extension of the elastic AC.

§§ 1—6. 2.

1. Some boys out for a paperchase arrive at a hill-top; from which they see three churches that they recognize as those in three villages A, B, C marked on the map they carry. They judge that they are equally distant from all three, and find from the map that AB = 10 miles, BC = 12 miles, CA = 17 miles. Shew by means of a figure drawn to scale (1 in. = 10 miles) how they determined their position, and give the distance of the hill-top from each village.

2. Draw an angle whose sine is ·6, and an angle whose cosine is $\frac{12}{13}$: construct an angle equal to the sum of these two angles and determine graphically its sine, cosine and tangent.

3. Find by geometrical constructions the values of

(i) $\sin(2\sin^{-1}\frac{3}{5})$;

(ii) $\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{12}{13}$;

(iii) $\sin^2(\tan^{-1}\frac{8}{15})+\cos^2(\tan^{-1}\frac{8}{15})$;

(iv) $\sin(\cos^{-1}c)$, $\cos(\sin^{-1}s)$, $\tan(\sin^{-1}s)$, $\sin(\tan^{-1}t)$, $\cos(\tan^{-1}t)$, $\tan(\cos^{-1}c)$.

4. Shew that the sine and cosine of any acute angle are each less than unity, whereas no matter what numerical value is assigned to the tangent the acute angle having that number for its tangent can be found.

§§ 6—8. 3.

1. Given *sin* A = ·263, obtain *graphically* the values, correct to three decimal places, of *cos* A, *tan* A, *cot* A, *sec* A, *cosec* A, taking 10 inches to represent the denominator of the fraction in each case and using a diagonal scale. Verify by decimal approximation that

(*a*) $\cos^2 A+\sin^2 A=1$,

(*b*) $\sec^2 A-\tan^2 A=1$,

(*c*) $\operatorname{cosec}^2 A-\cot^2 A=1$.

2. Solve the equation $2\sin\theta+\cos\theta=1$ by means of the identity $\sin^2\theta+\cos^2\theta=1$, for $\sin\theta$ and $\cos\theta$: find the value of θ from the tables.

3. Given $a\cos\theta+b\sin\theta=c=b\cos\theta-a\sin\theta$,

solve the equations for $\cos\theta$, $\sin\theta$; and by means of the identity $\cos^2\theta+\sin^2\theta=1$ shew that $a^2+b^2=2c^2$.

4. The following dimensions were taken from the end wall of a house:

height to eaves	37 feet,
height to ridge	51 feet,
breadth at eaves	23 feet;

if the two halves of the roof were of equal slope, find the angle of slope by calculation. Verify by a drawing to scale.

§§ 6—9. 4.

1. A tower stands on the top of a hill, whose side has a constant gradient from bottom to top. From the top of the tower the depressions of three points A, B, C which lie along the line of greatest slope through the foot of the tower are 34°, 36°, 40°. If $AB = BC = 100$ ft., find the height of the top of the tower above the level of A by a diagram drawn to scale. [$1'' = 25$ ft. requires a $12'' \times 18''$ sheet of paper.]

2. If $\tan A + \sec A = 2$, shew that $\sin A = \frac{3}{5}$, A being an acute angle; also shew that $\cot A + \operatorname{cosec} A = 3$.

3. Solve question 2 by constructing on the same diagram, graphs of

$$y = \tan A + \sec A, \quad y = \sin A, \quad y = \cot A + \operatorname{cosec} A$$

from the tables, on as large a scale as convenient.

4. The sides of a rhombus are each 5 inches long, and the length of one diagonal is 9 inches, find the length of the other diagonal and the acute angle between two adjacent sides.

§§ 7—10. 5.

1. The elevations of the top and bottom of a flag-staff supposed to be 32 ft. high standing at the edge of a perpendicular cliff, taken from a certain station on the level sands at the foot of the cliff, are 48° 50′, 36° 40′ respectively, and the height of the cliff is calculated from these data. If the real height of the flag-staff is 31 ft. 8 in., what is the error in the height of the cliff?

2. Prove the following statements

 (1) $(\sin A + \cos A)^2 = 1 + 2 \sin A \cos A$.

 (2) $\tan A + \cot A = \sec A \,.\, \operatorname{cosec} A$.

3. Solve the equations

 (1) $2\cos^2\theta - 7\cos\theta + 3$, for $\cos\theta$.

 (2) $12\tan^2\theta - 13\tan\theta + 3$, for $\tan\theta$,

and obtain a value of θ for each equation from the tables.

4. Plot the graphs of $y=\sin x$, $y=\cos x$, and obtain from them a graph of $y=\sin x+\cos x$ and $y=\cos x-\sin x$ for values of x between 0° and 90°.

5. Plot a graph of $s=10t+16t^2$; obtain the tangent of the angle of slope for successive values of t and plot a line to represent these values. What do the ordinates of points on this line represent?

§§ 7—12. 6.

1. Two adjacent sides of a parallelogram are 11, 17 inches respectively, and they include an angle of 37° 12′ 15″. Find the lengths of the two diagonals and the area.

2. (i) If $\cot A+\operatorname{cosec} A=5$, find $\cos A$;

(ii) if $\sec A=7$, find the value of $\dfrac{\sin A+\cos A}{\tan A+\cot A}$.

3. Find the values of $\dfrac{nr^2}{2}.\sin\dfrac{360^\circ}{n}$, when n has the values 11, 21, 31, and $r=10$ inches. [Note. These values are the areas of regular polygons of 11, 21, 31 sides respectively, inscribed in a circle whose radius is 10 inches.]

4. Find, by calculation from five-figure tables, the value of

$$\cos\frac{360^\circ}{7}+\cos\frac{720^\circ}{7}+\cos\frac{1080^\circ}{7}+\cos\frac{180^\circ}{7}\cos\frac{540^\circ}{7}\cos\frac{900^\circ}{7}.$$

5. The diameter of the base of a conical tent is 20 feet; the length of the narrow triangular strips of canvas, which are joined together to form the tent, is 17 ft. 6 in.; find the height of the tent.

§§ 7—13. 7.

1. A railway 200 miles long, which may be considered straight, crosses a country from sea to sea; the heights in feet above sea-

level which it reaches at the end of every 10 miles by the map are 500, 625, 670, 590, 542, 563, 700, 810, 920, 960, 978, 1020, 1033, 840, 706, 402, 383, 386, 394, sea-level. Find the tangent of the angle of slope for each section, as fractions with unit numerators.

2. Prove, by calculation from the tables, that

$$\sin^{-1}\frac{1}{3}+\sin^{-1}\frac{1}{3\sqrt{11}}+\sin^{-1}\frac{3}{\sqrt{11}}=90^\circ,$$

acute-angled values being taken for each $\sin^{-1}$.

3. From a ship sailing due N., two lighthouses are observed to bear N.E. and N.N.E. respectively: after the ship has sailed 20 miles the lighthouses are seen to be in a line due E. Find the distance between the lighthouses.

4. The ropes of a swing are 25 ft. long, the heights of the seat above the ground at its highest and lowest points are 14 ft. and 3 ft. respectively. What is the angle on each side of the vertical through which the swing oscillates?

5. ABC is a triangle, BD bisects the angle at A and meets BC in D. Prove $AB \cdot AC = AD^2 + BD \cdot DC$.

Hence find the angle BAC, if $AB=125$, $AC=80$, $AD=60$.

§§ 14—15. 8.

1. Prove the law for Multiplication of Logarithms directly from Definition (*b*) on p. 78, viz.:

$$a^{\log_a N} = N.$$

2. Find the values of the three fractions,

$$\text{(i) } \frac{1{\cdot}732\times4{\cdot}273}{8{\cdot}634};\quad \text{(ii) } \frac{1{\cdot}732\times8{\cdot}634}{4{\cdot}273};\quad \text{(iii) } \frac{4{\cdot}273\times8{\cdot}634}{1{\cdot}732};$$

(*a*) by extracting from the five-figure tables on pages 88, 89, the *three-figure* logarithms of 17, 18, 42, 43, 86, 87, etc.:

(*b*) by extracting from the same tables, the *four-figure* logarithms of 173, 174, 427, 428, 863, 864, etc.:

(*c*) by using the *five-figure* logarithms given in the tables:

and find the percentage error by comparison with the answers, which have been calculated with *seven-figure* tables.

3. Shew that (i) $\log_a b \times \log_b c = \log_a c$,

(ii) $\log_a b \times \log_b a = 1$.

4. Find the values of

$$\log_2 0{\cdot}5,\quad \log_{0\cdot5} 2,\quad \log_8 \sqrt{2},\quad \log_{\sqrt{2}} \sqrt[3]{16},\quad \log_2 10.$$

5. Shew by means of the identity $\log_a b \times \log_b c = \log_a c$ that if a set of tables of logarithms to any base have been calculated, then a set to any other base can be obtained, by multiplying each logarithm in the first set by a constant factor.

§§ 14—17. 9.

1. Find by logarithms the value of

$$pv^{\gamma},$$

when $p = 726$, $v = 1{\cdot}77$, $\gamma = 1{\cdot}14$.

2. Tables of logarithms are calculated to the base e, where $e = 2{\cdot}71828$ approximately: find the constant factor or **modulus** necessary to transform the tables to the base 10.

3. Calculate the area of the triangle ABC from the formula

$$\Delta = \frac{c^2}{2} \cdot \frac{\sin A \sin B}{\sin C},$$

given $c = 750$ ft., $A = 46^\circ\, 27'\, 30''$, $C = 75^\circ\, 16'\, 45''$.

4. Find the percentage error in the last result if the two angles, owing to a faulty instrument, are each 15″ greater than the given observed values.

5. (*a*) In a triangle ABC shew that

$$(b+c) \sin \frac{A}{2} = a \sin \left(C + \frac{A}{2}\right),$$

and explain how to find the angles of a triangle, when an angle, the opposite side, and the sum of the other two sides are given.

(*b*) In the calculation you have to determine an angle from its sine; shew that, though this be the case, the data will give only one triangle.

(*c*) Given that one angle of a triangle is 110°, the opposite side is 5,000 units, and the sum of the other two sides 6,000 units, find the remaining angles to the nearest minute.

§§ 15—19. 10.

1. If d_1, d_2, d_3 are the distances of the centre of the inscribed circle from the angular points A, B, C of the triangle, prove

$$\text{(i)}\quad ad_1^2+bd_2^2+cd_3^2=abc,$$

$$\text{(ii)}\quad \frac{d_1d_2d_3}{r}=\frac{2abc}{a+b+c}.$$

2. Given $\log 35=a$, $\log 325=b$, $\log 245=c$, find

$$\log 5,\quad \log 7,\quad \log 13.$$

3. If the triangle circumscribing an isosceles triangle is equal to the escribed circle touching one of the equal sides, the triangle is right-angled.

4. Two sides of a triangle are 71 ft. and 25 ft. respectively, the contained angle is 69° 32′; solve the triangle and calculate the error in the third side due to a slight error, say of 2 inches, in the length of the smaller of the two given sides, i.e. side = 25 ft. 2 in. really.

5. (*a*) Explain the principle of the vernier. If a graduated arc reads angles to 10′, how ought the vernier to be divided so that the arc may be made to give angles true to 10″?

(*b*) Draw a part of a scale to represent inches and tenths of an inch from 19 to 21 inches; and draw a vernier in contact with the scale and so placed that the reading may be 20·26 in.

APPENDIX.

INTRODUCTION TO THEORETICAL TRIGONOMETRY.

§ 20. Circular measure.

The units employed for measuring angles, already given in § 2, are not as a rule used in Theoretical Trigonometry. The *theoretical* unit in all branches of mathematics is usually taken to be what is called an **absolute unit**: this being chosen, so that the measure of the quantity is represented by 1, when the measures of the fundamental quantities, such as mass, length and time, are also represented by 1.

Thus in Mechanics, the British Absolute Unit of Force is that Force which, if it acted on a mass of *one* pound for *one* second, would produce in it a velocity of *one* foot in *one* second. This is not the practical unit, which is about 32 times as great, namely, the weight of one pound.

Similarly, since an angle is the measure of the rotation of a vector, the *absolute unit* for angles is taken as **the angle through which a vector of unit length has rotated round one fixed end, when its other end has described an arc of unit length.** In other words it is *the angle subtended at the centre of a circle whose radius is of unit length by an arc equal to the radius.*

It cannot be concluded as an *immediate* deduction from this definition that the unit angle is the angle subtended at the centre of any circle by an arc equal to the radius; nor will this unit of angle be of any service unless it can be shewn that

(i) For different lengths of the vector, the arc described by the free end is proportional to the vector for the same angle.

(ii) For the same length of vector, the arc described by the free end is proportional to the angle.

In addition it is advisable that the relations between the absolute unit and the practical units, the right angle and the degree, should be known numbers; in other words it must be shewn that

(iii) The ratio of the circumference of a circle to its diameter is constant for all circles.

In modern text-books on Geometry, several practical methods for finding the approximate length of any curved line are given: but for circular arcs a *theoretical* method, which may be deduced from the following experiment, is generally adopted.

78. Set four stout pins firmly in a drawing board, fasten a piece of string* to A and pass it round B, C, and D and strain it tight enough to make the parts AB, BC, CD perfectly straight. Set a cylindrical piece of wood, such as are used for supports for physical or chemical apparatus (the cover of a round tin will do) in the position shewn in Fig. 73, arranging the pin B so that the parts BA, BC of the string just fail to touch the arc AC; that is BA and BC are tangents to the circular arc AC.

(i) Make a mark with a pen or pencil on the string where it touches the pin D.

* A tape measure substituted for the string is an improvement.

(ii) Take out the pin B, *pull* the free end of the string beyond D until the part between A and C fits tightly round the arc AC, and again make a mark on the string where it touches the pin D.

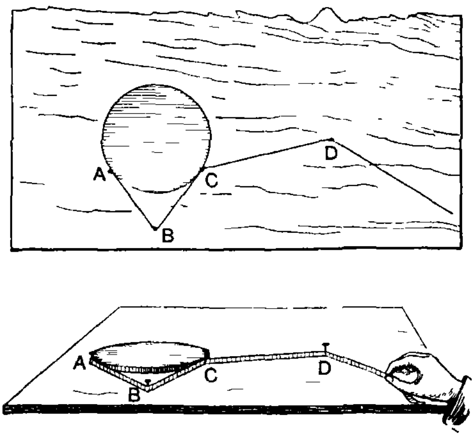


Fig. 73.

(iii) Take away the piece of wood or tin cover and again *pull* the string until the string between A and C is tight, i.e. straight, and again make a mark on the string where it touches the point D.

Since in Expt. 78, the free end of the string was lengthened by the *pull* at each change of the conditions of the experiment, (i), (ii) and (iii), the following conclusion is obvious:

The length of an arc is intermediate between the sum of the lengths of the tangents at its extremities and the length of the chord of the arc.

79. Take any arc AC, draw the tangents at A and C, join AC: measure AC and AB + BC. Then the length of arc AC lies between the lengths of AC and AB + BC.

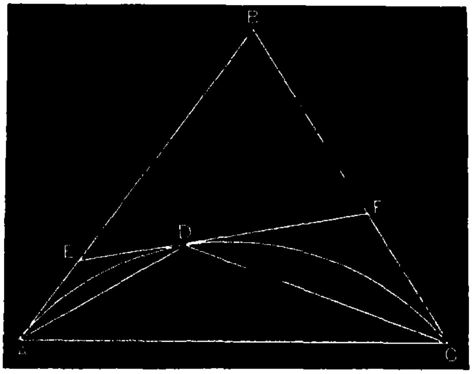


FIG. 74.

Take a point D on the arc and draw EDF to touch the circle at D and intersect AB, BC in E and F respectively; join AD, DC: measure AD + DC and AE + EF + FC. The length of the arc lies between these lengths. Also

$$AD + DC > AC \text{ and } AE + EF + FC < AB + BC.$$

Hence both the limits between which the length of the arc must lie are closer approximations than before.

Take two more points, one on the arc AD, the other on the arc DC, draw the corresponding tangents and chords, and obtain a third approximation, closer still, to the limits between which the length of the arc must lie.

Carry on the experiment as far as possible and tabulate the results as in the table below, which was obtained for an arc of $114\frac{1}{2}°$ in a circle of $7\frac{1}{2}$ in. radius, the last result having been obtained by "stepping out" the arc with the divider points at $\frac{1}{4}$-inch apart.

	Sum of tangents	Sum of chords	Intermediate points
1	23·50	12·65	0
2	16·75	14·30	1
3	15·45	14·85	3
4	15·05	14·95	8
	—	15·00	59

The results tabulated above may be shewn graphically, when rate of effect produced per extra intermediate point is seen at a glance.

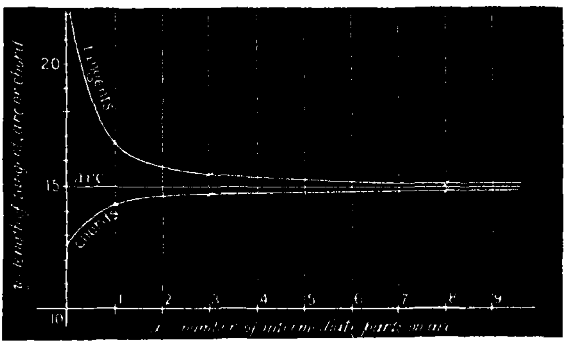


Fig. 75.

For the purpose of the experiments which follow, the student may *assume* that the length of an arc is very approximately given by carefully stepping it out with the points of the dividers at $\frac{1}{10}$-inch apart, the error in defect being less than ·1 per cent. of the total length of the arc, for any circle whose radius is greater than one inch.

80. Draw any acute angle BAC: with centre A draw arcs P_1Q_1, P_2Q_2, ... Measure the several arcs (PQ), the corresponding radii (AP) and obtain as a decimal the ratio $\left(\frac{PQ}{AP}\right)$, and exhibit the result in tabular form.

Repeat the experiment for an obtuse angle.

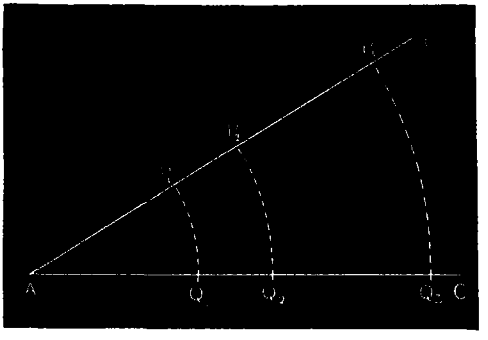


FIG. 76.

If the measurement of the arcs, by "stepping out" with the divider points at a distance of $\frac{1}{10}$-inch apart, is carefully performed, any excess beyond $\frac{1}{10}$-inch, being either estimated by eye or measured on a diagonal scale reading to $\frac{1}{100}$-inch, the results in the column for the ratio should be the same to the second or third place of decimals. Deduce the following theorem:

For a given angle at the centre of a system of concentric circles, the subtending arcs are proportional to the radii.

81. Draw a circle of any conveniently large radius. Draw several radii $OXOA_1$, OA_2, OA_3.... Measure the arcs XA_1, XA_2, XA_3..., and the $\angle$s XOA_1, XOA_2, XOA_3..., with a protractor, and deduce the following theorem:

In any given circle, angles at the centre of the circle vary as the arcs on which they stand.

The theorem deduced in Expt. 81 follows naturally from the theorem, proved on page 5, that equal arcs of a circle subtended equal angles at the centre.

For, let O be the centre of a circle, and AOB, BOC, two angles at the centre standing on the arcs AB, BC; let XY be an arc, which is contained p times in AB and q times in BC.

Divide AB into p equal arcs, and BC into q equal arcs each equal to XY.

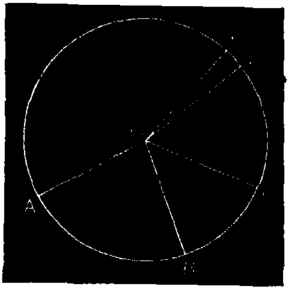


FIG. 77.

Then each of these arcs will subtend at O a number of equal angles, each equal to XOY, of which AOB will contain p, and BOC will contain q.

$$\therefore \quad \frac{\angle \mathrm{AOB}}{\angle \mathrm{BOC}} = \frac{p}{q} = \frac{\mathrm{AB}}{\mathrm{BC}}.$$

If s, a, θ, be the measures of the arc, the radius, and the angle subtended by the arc at the centre, of any circle, it has been shewn that

(i) s varies as a, when θ is constant;

(ii) s varies as θ, when a is constant.

Hence, by the theory of variation (see Algebra)

s varies as $a\theta$ when both a and θ vary.

Or symbolically

$$s = c_1 . a \ (\theta = \text{const.}),$$
$$s = c_2 . \theta \ (a = \text{const.}),$$
$$\therefore \quad s = c . a . \theta,$$

where c_1, c_2, c are constants.

If, then, we choose as unit angle, that angle which is subtended at the centre of a circle by an arc equal to the radius, we have in the formula $s = c.a.\theta$,

$$s = a \text{ when } \theta = 1,$$

$$\therefore \quad c = 1,$$

and the formula becomes $s = a\theta$; i.e., the arc measures the angle on a scale whose unit equals the radius.

82. Draw circles of radii, 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, 5 inches respectively; measure the circumferences by "stepping out" with the points of the dividers at $\frac{1}{10}$-inch apart; obtain the ratio $\frac{\text{circumference}}{\text{diameter}}$, and tabulate the results.

Deduce that **in any circle, the ratio of its circumference to its diameter is constant.**

If the measurement is carefully done, the results in the last column should agree to at least two places of decimals. The correct answer true to two places of decimals is 3·14. This ratio or abstract number, for which the letter π is used, may also be determined in a manner similar to that in Expt. 79, by using the fact that the circumference is intermediate between the perimeters of regular polygons inscribed in and circumscribed about the circle.

83. Let A, B, C..., D, E, F, G... be the angular points of regular polygons of n sides, inscribed in and circumscribed about a given circle, whose centre is O, and whose radius is r.

Join OA, OB, OC, OE, OF: shew that

Fig. 78.

(1) perimeter of inscribed polygon $= n \,.\, \text{AB} = 2nr \,.\, \sin\frac{360^\circ}{2n}$,

(2) ,, circumscribed ,, $= n \,.\, \text{EF} = 2nr \,.\, \tan\frac{360^\circ}{2n}$.

Using the formulae obtained for the perimeters of inscribed and circumscribed regular polygons of n sides, the following table may be constructed from tables of natural sines and tangents, for a circle whose radius is $\frac{1}{2}$-inch and whose circumference is therefore π inches.

n	$2nr \sin \frac{360°}{n}$	$2nr \tan \frac{360°}{2n}$	n	$2nr \sin \frac{360°}{n}$	$2nr \tan \frac{360°}{2n}$
5	2·93895	3·63270	36	3·13776	3·15364
6	3·00000	3·46410	40	3·13840	3·14800
8	3·06144	3·31368	60	3·14040	3·14460
10	3·09020	3·24920	72	3·14064	3·14352
12	3·10584	3·21540	90	3·14100	3·14280
15	3·11865	3·18840	120	3·1412280	3·1423080
18	3·12570	3·17374	180	3·1414320	3·1419180
20	3·12860	3·16760	360	3·1415400	3·1416840
24	3·13272	3·15960	720	3·1415760	3·1416480

The results in the foregoing table for the perimeters of the polygons are correct to within ·0001 for $n = 5$ to $n = 20$, to within ·0005 for $n = 20$ to $n = 90$, and to within ·00003 for $n = 90$ to $n = 720$, the last four results being obtained from seven-figure tables. By considering the differences in these last four it is evident that the perimeter of the circumscribed polygon shows a rate of decrease which is greater than the rate of increase of the perimeter of the inscribed polygon. Hence the circumference of the circle is about 3·1416.

It has been deduced from the results obtained in Expt. **82** that the ratio of the circumference of any circle to its diameter is constant for all circles. This can be theoretically proved as follows.

N.B. *It is assumed that, if* AB, AC *are two tangents to a circle, then* $AB+AC>$ **arc** $BC>$ **chord** BC. [Cp. p. 147.

Suppose LA, AM are two sides of a regular polygon of n sides circumscribed about the circle, touching the circle at their middle points B, C: then BC is a side of a regular polygon of n sides inscribed in the circle.

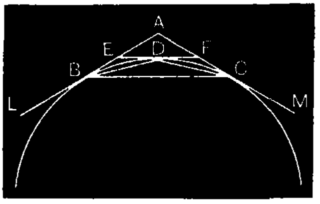


FIG. 79.

Bisect the arc BC at D and draw EDF tangent to the circle at D: then it will be seen, by symmetry, that EF is the side of a regular polygon of $2n$ sides circumscribed to the circle, and BD is a side of a regular polygon of $2n$ sides inscribed in the circle.

Again, since the sides of a circumscribed polygon are bisected at their points of contact with the circle,

the perimeter of the circumscribed n-gon $=n(AB+AC)$,

" " " $2n$-gon $=2n\,.\,EF$
$=n[BE+EF+FC]$,

the perimeter of the inscribed n-gon $=n\,.\,BC$,

" " " $2n$-gon $=2n\,.\,BD$
$=n\,.\,[BD+DC]$.

But, since two sides of a triangle are greater than the third, it is clear that

$$BD+DC>BC,$$

and $AB+AC>BE+EF+FC$.

Hence, if the number of sides be doubled, the perimeters of regular polygons circumscribing the circle are diminished, and the perimeters of regular polygons inscribed in the circle are increased.

But it follows, from the assumption made above, that if ${}_nP_c$, S, ${}_nP_i$ are the perimeter of a circumscribed n-gon, the circumference of the circle, the perimeter of an inscribed n-gon respectively,

$${}_nP_c>{}_{2n}P_c>S>{}_{2n}P_i>{}_nP_i.$$

$$\text{Hence, if} \quad S = {}_nP_c - d_c, \qquad \text{or} \qquad S = {}_nP_i + d_i,$$

d_c, d_i are diminished indefinitely as the number of sides become $2n$, $4n$, $8n$, etc., and can be made smaller than any assignable quantity by taking the number of sides large enough. **Hence the circumference of a circle may be considered as equal to the perimeter of a circumscribing (or of an inscribed) polygon, the number of whose sides is infinitely great.**

Now, suppose ABCD..., A′B′C′D′... are any two circles, placed so that their centres coincide at O.

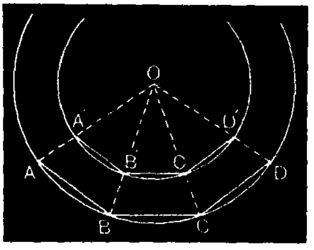


Fig. 80.

Let A, B, C, D ... be the vertices of a regular polygon of n sides inscribed in the greater circle. Join OA, OB, OC, OD ..., cutting the smaller circle in A′, B′, C′, D′ respectively.

Then A′, B′, C′, D′ ... will be the vertices of a similar and similarly situated polygon; i.e. a regular polygon of n sides, each side being respectively parallel to a side of the polygon ABCD....

Hence, ∵ $\triangle$s OBC, OB′C′ are equiangular,

$$\therefore \quad \frac{BC}{OB} = \frac{B'C'}{OB'},$$

$$\therefore \quad \frac{\text{perimeter of ABCD} \ldots}{OB} = \frac{\text{perimeter of A}'\text{B}'\text{C}'\text{D}' \ldots}{OB'};$$

this is true, no matter how great the number of sides of the polygons may be.

It is therefore true when the number is infinitely great; that is, it is true when the perimeters are equal to the circumferences of the circles respectively,

$$\frac{\text{circumference of} \odot \text{ABCD}}{OB} = \frac{\text{circumference of} \odot \text{A}'\text{B}'\text{C}'\text{D}'}{OB'}.$$

The true value of π has been worked out to several hundred decimal places by methods of more advanced Trigonometry, and it has been proved that it is a never-ending, non-recurring decimal, i.e. an **incommensurable** number. As far as the tenth decimal place,

$$\pi = 3{\cdot}1415926536.$$

Several approximations are used :—

(1) $\frac{22}{7}$ or $3\frac{1}{7} = 3{\cdot}\dot{1}4285\dot{7}$ for rough calculations;

(2) $3{\cdot}1416$ (divisible by 3, 7, 8, 11, 17), and

(3) $\frac{355}{113} = 3{\cdot}1415929\ldots$ for more exact work.

The measure of an angle in absolute units is called the **circular measure of the angle.** To avoid speaking of "an angle whose circular measure is n," the suggestive name **radian** was invented for the unit: thus we *may* speak of "an angle of n radians," instead of "an angle whose circular measure is n," just as we speak of "an angle of x degrees."

It follows at once from what has already been proved in this section that the circular measure of an angle varies as the number of degrees in the angle, and as soon as the angle under consideration has been expressed in degrees, and a fraction of a degree, its circular measure can be deduced by simple proportion, and *vice versa,* from the fundamental relation between the two measures of an angle of two right angles.

The circular measure of two right angles, by using the formula $s = a\theta$, is found to be $\dfrac{\text{semi-circumference}}{\text{radius}}$; that is, π.

$$\therefore\ \mathbf{180\ degrees} \equiv \pi\ \mathbf{radians.}$$

84. Find the circular measure of an angle of $43° 20' 30''$.

The circular measure of $180°$ is π,

$$\therefore \quad " \quad " \quad 1° \text{ is } \frac{\pi}{180},$$

$$\therefore \quad " \quad " \quad 43° \tfrac{41}{120} \text{ is } \frac{\pi}{180} \times 43 \tfrac{41}{120}$$

$$= \frac{355 \times 520 \cdot 1}{113 \times 180 \times 120}$$

$$= \cdot 75645.$$

85. Find the number of degrees, minutes, and seconds (i.e. the sexagesimal measure) of the unit of circular measure.

Since π radians contain $180°$,

$$\therefore \text{ unit of circular measure} = \frac{180°}{\pi}$$

$$= 180° \times \frac{113}{355}$$

$$= 57° 17' 45'' \text{ nearly.}$$

1. *Find the number of degrees, minutes and seconds in an angle* whose circular measure is $2 \cdot 5$.

2. Find the circular measure of an angle of $36° 15' 12''$.

3. If the radius of a circle is 10,000 ft., find to the nearest foot the length of an arc which subtends an angle of one minute at the centre.

4. The distance between successive graduations on the outer rim of a graduated circle is one-fiftieth of an inch, and this interval subtends an angle of one minute at the centre of the circle; find the diameter of the circle to the nearest inch.

5. A man runs round a circular track at 12 miles an hour. The circular measure of the angle through which his body has turned is $1 \cdot 5$ at the end of a minute; find the diameter of the track.

6. How long does it take the hour-hand of a watch to rotate through the angle whose circular measure is $\frac{\pi}{15}$?

The area of a circle can be obtained by a method similar to that by which the length of the circumference was found in Expt. **83.** For the area is intermediate between those of the inscribed and circumscribed regular polygons of n sides.

From Fig. 78 on p. 152, it follows that

(1) Area of inscribed polygon $= \frac{1}{2}nr^2 \sin \frac{360^\circ}{n}$.

(2) " circumscribed " $= nr^2 \tan \frac{360^\circ}{2n}$.

If we calculate the values of $\frac{1}{2}n \sin \frac{360^\circ}{n}$ and $n \tan \frac{360^\circ}{2n}$, we obtain

n	$\frac{1}{2}n \sin \frac{360^\circ}{n}$	$n \tan \frac{360^\circ}{2n}$	n	$\frac{1}{2}n \sin \frac{360^\circ}{n}$	$n \tan \frac{360^\circ}{2n}$
5	2·37765	3·63270	36	3·12570	3·15364
6	2·59806	3·46410	40	3·12860	3·14800
8	2·82844	3·31368	60	3·13590	3·14460
10	2·93895	3·24920	72	3·13776	3 14352
12	3·00000	3·21540	90	3·13920	3·14280
15	3·05055	3·18840	120	3·1401600	3·1423080
18	3·07818	3·17374	180	3·1409550	3·1418180
20	3·09020	3·16760	360	3·1414320	3·1416840
24	3·10584	3·15960	720	3·1415400	3·1416480

with the same limits of accuracy as in the table for perimeters on p. 153: from which it is easy to deduce that

the area of a circle of radius $\mathbf{r} = \pi \mathbf{r}^2$.

Again, $\because$ area of circle $= \pi r^2 = r \, . \, \pi r$,

and semi-circumference $= \pi r$ by Expt. **83**,

$\therefore$ **the area of a circle is equal to that of a rectangle whose adjacent sides are equal to the radius and the semi-circumference respectively.**

This can be shewn practically by a method of dissection, as follows:

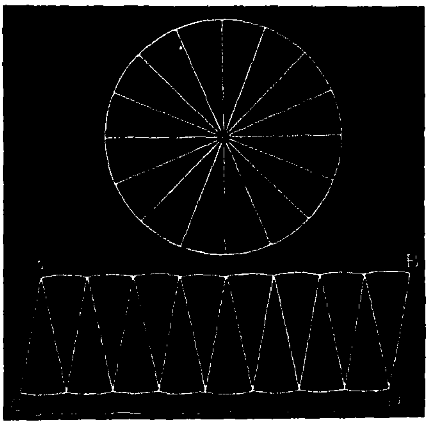


FIG. 81.

Draw a circle of any radius on paper and cut it out: divide it into a number of equal parts as in Fig. 81. Rearrange the parts, as in the lower diagram ABCD. Note that if the number of pieces is increased the broken lines AB, CD become more and more nearly straight: and ultimately, when the number of pieces is very great, the curves disappear and the lines AB, CD become straight lines equal in length to the semi-circumference of the circle.

Moreover the angles at D and B become right angles.

Hence the figure becomes ultimately a rectangle whose adjacent sides are respectively equal to the radius and semi-circumference of the given circle.

EXERCISES.

1. The driving wheel of a locomotive is 7 ft. 6 in. high: how many revolutions per minute does it make when the speed of the train is 60 miles per hour, supposing that there is no slipping between the wheel and the rail?

2. Express in circular measure the interior angles of regular polygons of 5, 6, 8, 10 sides.

3. Two pulleys of radii 3 in. and 7 in., are set on short parallel axes projecting at right angles from a vertical wall at a distance 12 in. apart. Find, to the nearest inch, the length of the belt which will go round these pulleys, so that they rotate in the same direction, i.e. the belt does not cross between the pulleys.

4. Find the length of the belt if the pulleys in Ex. 3 have to rotate in opposite directions, i.e. the belt crosses between the pulleys.

5. An equilateral triangle is inscribed in a circle, and a regular hexagon is circumscribed to the same circle: find the ratio of their areas.

6. The areas of regular polygons, of the same number of sides, inscribed in and circumscribed to a circle, are in the ratio 3 : 4. Find the number of sides.

7. Give a formula to find the sectional area of a cylinder, the circumference only being given. What must be the girth of a water pipe, made of iron $\frac{1}{2}$-inch thick, to give an effective sectional water area of one square foot?

8. A florist exhibiting a group of plants at a show wishes to arrange them with the tallest in the middle in the form of a circular cone: each competitor is allowed 50 sq. ft. of floor for his collection: find the length of the string which he must use as a radius to mark out the circle on the floor.

9. A circle of 10″ radius is divided into three equal parts by the circumferences of two concentric circles: find the radii of the two circles.

10. How much will it cost to turf a circular lawn, whose diameter is 35 yds., at 6*d.* a sq. yd.?

§ 21. Use of negative sign in Graphs.

In Geometry and Graphical Algebra it is very often necessary to consider *direction* as well as *magnitude* in determining the length of a straight line. Thus, if A and B are two points, the distance of A and B (say, two miles *North-West*) is considered to be different from the distance of B from A (two miles *South-East*). The most convenient method of indicating opposite directions along a straight line is by the use of the algebraical signs + and −; that direction which is most suitable in each particular case being *chosen* for the positive direction of measurement. Thus, if we consider South-East to North-West as positive, the distance of A from B, called *the length of* BA is + 2 miles, and the distance of B from A, *the length of* AB, is − 2 miles.

Similarly, the two possible directions of rotation in describing angles are distinguished from one another by choosing one direction as the positive direction, the other direction being considered negative. The usual convention is to reckon as **positive** angles described by an **anti-clockwise** rotation, i.e. in a direction contrary to that in which the hands of a watch move.

When account is thus taken of the direction of measurement, or **sense**, of a line or angle, some **zero position** must be chosen. This, for measurement of lines, is called the **origin** or **pole**; and, for measurement of angles, the **initial line**. The origin and initial line are chosen to suit the needs of each particular case.

86. Take a point O as origin on any straight line and mark the points A, B, C, D, E so that $OA=3$, $AB=5$, $BC=-6$, $CD=2$, $DE=-9$. Write down the lengths of BA, BD, EO; and verify that

(1) $OA+AB+BC+CD+DE=3+5-6+2-9=-5=OE$.

(2) $AB+BC+CA=0$.

To fix the position of a point on a certain surface, the usual method is to find two lines, *which can easily be determined*, intersecting in the point. Thus, (1) places on the earth's surface are fixed by the latitude and longitude; (2) the usual method of piloting a ship along a difficult channel, such as the entrance to Liverpool, is to keep two landmarks, beacons, or buoys in a line until two others are in a line, the ship being then at the point of intersection of the two lines, which indicates the point at which the course has to be changed; (3) buried treasure can be hidden and its position found again by the intersection of two circles whose radii are known; for this two origins or poles are required, such as the distances from a tree and a telegraph post, and also directions for distinguishing the particular tree and post; (4) the distance and bearing of one point from another give a circle of known radius and a straight line making a known angle with a fixed direction, say North. If, however, two circles are used as in (3), they intersect in two points and it is necessary to have some other thing to determine which of the two points is the right one. With a circle and a straight line drawn from the centre, as in (4), although they intersect in two points, the point can be at once determined if we have some convention for an angle similar to that which distinguishes E.N.E. from E.S.E., which both make angles of $22\frac{1}{2}°$ with E., i.e., the idea of positive and negative angles: this, one of the most important methods of fixing a point, will be again referred to in § 23.

Another method, the most generally used, is that in which the position of the point is determined as the intersection of two straight lines parallel to two fixed straight lines, the distance of each line from the initial line, to which it is parallel, being measured *parallel to the other line.* This is the method of "squared-paper" work, though it is not necessary that the initial lines should be at right angles.

87. Draw any two straight lines OX, OY; produce them backwards through O to X′, Y′.

For distances along straight lines parallel to X′X or OX, consider those as positive that are drawn in the same sense as OX, and as negative, those in the opposite sense: for distances along straight lines parallel to Y′Y or OY, consider those as positive which are drawn in the same sense as OY and as negative, those in the opposite sense mark the points A, B, C, D, E, F, G, H from the following measurements:

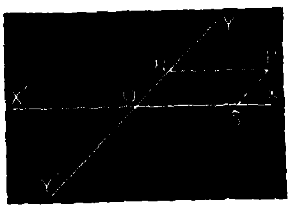


FIG. 82.

	A	B	C	D	E	F	G	H
Distance from OY parallel to OX	3	2	4	−5	0	−3	4	0
Distance from OX parallel to OY	2	3	0	4	−6	−2	−3	0

The lines X′OX, Y′OY are called **axes of coordinates** and the point O the **origin.**

If RP and SP denote the distances, measured in the senses of OX, OY respectively, which fix the position of the point P, RP and SP are called the ***Cartesian Coordinates of the point** P, or simply, the **coordinates of** P.

* *After* Des Cartes, the great mathematician.

In finding a point P, however, it is in general more convenient, since **OS** is equal to **RP**, to mark off a distance **OS** *along* **OX** and then the distance **SP** *parallel to* **OY**, or **OR**, **OS** may both be marked off along the axes and the parallelogram **SORP** be completed.

If the straight lines **X′OX**, **Y′OY** are taken at right angles, the parallelogram becomes a rectangle; the axes are then called **rectangular axes**, **SP** the **ordinate**, and **OS** the **abscissa**. When these lengths are given it is most important to know the signs of the lengths and also which is mentioned first.

88. A small chart is discovered containing a rough sketch, with directions, of the position of buried treasure.

The place is easily recognised by the discoverer of the chart by the four trees which are each 30 feet from the intersection of the hedges.

If the angle between the hedges is 65°, draw a diagram showing the *eight different positions* where the treasure might lie.

FIG. 83.

The abscissa is generally referred to as the x-coordinate, and the ordinate as the y-coordinate of a point; and if it is agreed that the x-coordinate shall always be mentioned first, a point **P** whose coordinates are

$$\left.\begin{array}{l} x=3 \\ y=4 \end{array}\right\},$$

can be referred to as the point "**P** [3, 4]" or, simply, as the point "(3, 4)."

89. Take rectangular axes and mark the position of the points

A[3, 4], B[4, −5], C[−5, −1], D[−1, 3]

using $\frac{1}{2}$ in. as the unit.

Calculate, by Pythagoras' Theorem, the length of AB, BC, CD, DA, AC ; verify by direct measurement.

It should be carefully observed what

$$\left.\begin{aligned} x &= 3 \\ y &= 4 \end{aligned}\right\},$$

as the coordinates of a point P, really mean. If a series of points, for which the distance from the y-axis for each is equal to $+3$, are plotted, these points all lie on a *straight line* parallel to the y-axis, and $x = 3$ is called the **equation of the line.** Similarly $y = 4$ is the equation of a line parallel to the x-axis, the distance of each point on it from the x-axis being equal to $+4$. The point P is the intersection of these lines, the only point for which $x = 3$ and $y = 4$.

Similarly $x = y$ is the equation of the straight line, each point of which is equidistant from the two axes. This is the straight line, *produced both ways through* O, which bisects the angle XOY.

90. Plot the line whose equation is $x = -y$, i.e. $x + y = 0$.

§ 22. Angles of any magnitude.

Whilst in surveying and for most, but not all, practical purposes it is only necessary to consider angles less than two right angles, as given by the Euclidean definition; in theoretical trigonometry no such limitation is made to the magnitude of an angle. An angle is defined as the *result of a certain operation*, viz. rotation.

DEF. When a straight line rotates in a plane about some point in its length from any one position to any other position it is said to have described an angle: the amount of rotation measures the angle. [Cf. p. 152.]

With this definition, given only the initial and final positions of the rotating line, it is impossible to fix the absolute magnitude of the trigonometrical angle. For it is not known (i) how many complete revolutions have been made, nor (ii) in which direction the rotation has taken place.

Thus even supposing, in Fig. 84, that no complete revolution has been made, the amount of rotation round O by which a line may be brought from the position OA to the position OB is obviously different for the two directions denoted by the arrow heads. If the circular measure of the positive angle represented by an arrow in the figure is θ, and that of the negative angle is $-\phi$, then

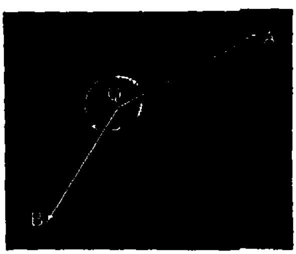


FIG. 84.

$$\theta + \phi = 2\pi,$$

since 2π is the circular measure of 4 rt. angles.

Now all the positive angles that could have been described with the initial and final positions of the rotating line as in the figure are contained in the series

$$\theta,\ \theta+2\pi,\ \theta+4\pi,\ \theta+6\pi,\ldots\ldots\ldots;$$

and all the negative angles are included in the series

$$-\phi,\quad -\phi-2\pi,\ -\phi-4\pi,\ -\phi-6\pi,\ldots\ldots\ldots,$$

i.e. $\theta-2\pi,\quad \theta-4\pi,\quad \theta-6\pi,\ \ldots\ldots\ldots\ldots;$

Hence all the angles which are determined by the inclination of the two straight lines **OA**, **OB** are included in the formula

$$\theta+2n\pi,$$

where n is any integer positive or negative.

The angle between any two non-intersecting straight lines is equal to the angle between any two straight lines drawn parallel to and in the same sense as the given straight lines.

In practice, one of the given straight lines is usually considered as the initial line, and a straight line is drawn intersecting it which is parallel to and in the same sense as the second of the given straight lines.

Thus in Fig. 85, we consider **OA** as the initial line and draw **OD** parallel to and in the same sense as **PQ**.

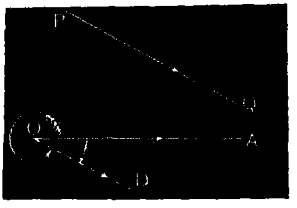


FIG. 85.

Then the angle between **PQ** and **OA** is the angle **AOD**, either the big positive angle, or the small negative angle—marked in Fig. 85 with arrows—or these angles increased or diminished by some multiple of four right angles.

It is important to remember the exact way in which such a line as OD is drawn :—

Through the point O (*the point of* departure *in* OA), OD *is drawn parallel to and in the same sense as* PQ.

Exercises.

1. Mark with arrows the positive and negative angles less than four right angles between the straight lines (Fig. 85).

    (i) PQ and AO,

    (ii) QP and OA,

    (iii) QP and AO,

using OA as the initial line.

2. Repeat Ex. 1 using PQ as the initial line.

3. Draw figures showing angles of :—

    (i) $270°$, (ii) $-235°$, (iii) $1000°$, (iv) $-625°$;

give the smallest positive and the smallest negative angles for each case, which the positions of the arms of the angles might determine.

4. Draw any irregular pentagon on a large scale, cut it out and tear off the corners; arrange the angles round a point so as to shew that their sum is six right angles.

5. Repeat Ex. 4 for the case of irregular or regular rectilinear figures of 6, 7, 8, 10 sides respectively, and verify the formula: The sum of all the interior angles of a polygon of n sides is equal to $2n-4$ right angles.

§ 23. Trigonometrical ratios for angles of any magnitude.

Let a straight line **OP**, rotate round **O** through an angle whose circular measure is θ. Let its initial position coincide with **OX**, and its final position with **OP** in one of the four diagrams in Fig. 86, where a positive value

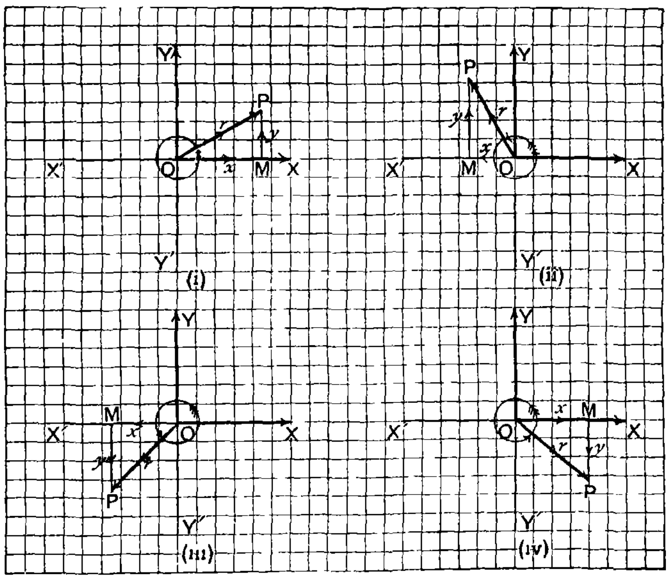


Fig. 86.

has been indicated for θ, by the arrows, although what follows is also true if the position **OP** had been reached by a negative rotation.

If r is the length of OP, it was shewn in § 21 that r and θ determined the position of the point P: r and θ are called the **polar coordinates** of P referred to O as pole and OX as the initial line. Let OY make with OX an angle equal to $+90°$ or $+\frac{\pi}{2}$ (the circular measure of 90°): let x and y be the cartesian coordinates of P. Then the general values of $\sin\theta$, $\cos\theta$ and $\tan\theta$ may be thus **defined.**

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ \tan\theta = \frac{y}{x},$$

where x and y are taken positive when measured in the same sense as OX and OY respectively, and r, which is really $\pm\sqrt{x^2+y^2}$, is taken as positive when measured outwards from the origin along the radius vector.

These ratios may also be defined in terms of the sides of the triangle OPM in Fig. 86, as

$$\sin\theta = \frac{\mathrm{MP}}{\mathrm{OP}},\ \cos\theta = \frac{\mathrm{OM}}{\mathrm{OP}},\ \tan\theta = \frac{\mathrm{MP}}{\mathrm{OM}},$$

if the order of the letters, determining the sign of the line, is carefully noted. In the figures arrow-heads are placed on the ends of the lines OX, OY to shew the directions of positive measurements for x and y respectively, and the arrow-heads on the sides are shewn pointing **from O to M, and from M to P.** The agreement (or otherwise) in direction between these latter and those on the lines **OX,** OY denoting the directions of positive measurement at once determine the signs of the ratios.

91. Find the value of tan 150° or $\tan\frac{5\pi}{6}$.

It is evident from Fig. 87, that **MP** is positive and **OM** is negative;

$$\therefore \tan\frac{5\pi}{6} = \frac{\mathsf{MP}}{\mathsf{OM}} = \text{a negative quantity.}$$

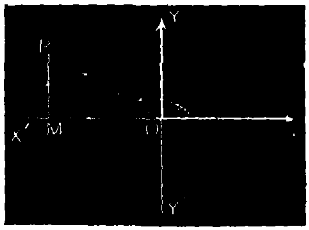


Fig. 87.

Again the Δ **OPM** is half an equilateral triangle whose sides are equal to **OP**; hence, if **OP**$=r$, the lengths of the straight lines **MP, OM** (*without regard to sign*) are $\frac{r}{2}$, $\frac{\sqrt{3}r}{2}$ respectively.

$$\therefore \text{ arithmetical value of } \frac{\mathsf{MP}}{\mathsf{OM}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 0{\cdot}57735.$$

$$\therefore \tan\frac{5\pi}{6} = -0{\cdot}57735.$$

92. Find in a similar manner the values of

(1) $\sin 240°$, (2) $\tan\frac{5\pi}{4}$, (3) $\cos(-135°)$,

(4) $\cos(-1110°)$, (5) $\cos\pi$, (6) $\tan\frac{\pi}{2}$.

§ 24. Graphs of the Trigonometrical Ratios for angles of any magnitude.

It is plain from § 23 that the sign and magnitude of the trigonometrical ratios of an angle are always the same when the rotating line occupies the same position, no matter how many complete revolutions it has made, in either the positive or negative direction of rotation: for the ratios only depend on the final *position* of the rotating line. Also no matter what was the initial magnitude of an angle A, when the rotating line has made one complete revolution either positively or negatively, and again reaches the position which determined the angle A, each of the ratios has passed through every possible value both as regards sign and magnitude: and if the rotation still goes on each of the ratios will *repeat* the same set of all possible values, in exactly the same order, for every complete revolution.

When one quantity depends on a second quantity in such a way that as *successive* values are given to the second, the first repeats a certain set of all possible values in exactly the same order, the first quantity is said to be a *periodic function* of the second.

Thus $\sin \mathsf{A}$, $\cos \mathsf{A}$, $\tan \mathsf{A}$ are periodic functions of A.

Again, since the other ratios are *defined* by the relations

$$(1) \quad \operatorname{cosec} \mathsf{A} \sin \mathsf{A} = 1,$$

$$(2) \quad \sec \mathsf{A} \cos \mathsf{A} = 1,$$

$$(3) \quad \cot \mathsf{A} \tan \mathsf{A} = 1,$$

it follows that $\operatorname{cosec} \mathsf{A}$, $\sec \mathsf{A}$, $\cot \mathsf{A}$ are also periodic functions of A. It should also be observed that the *cosec* and *sin*, the *sec* and *cos*, the *cot* and *tan* have their signs respectively equal for all angles.

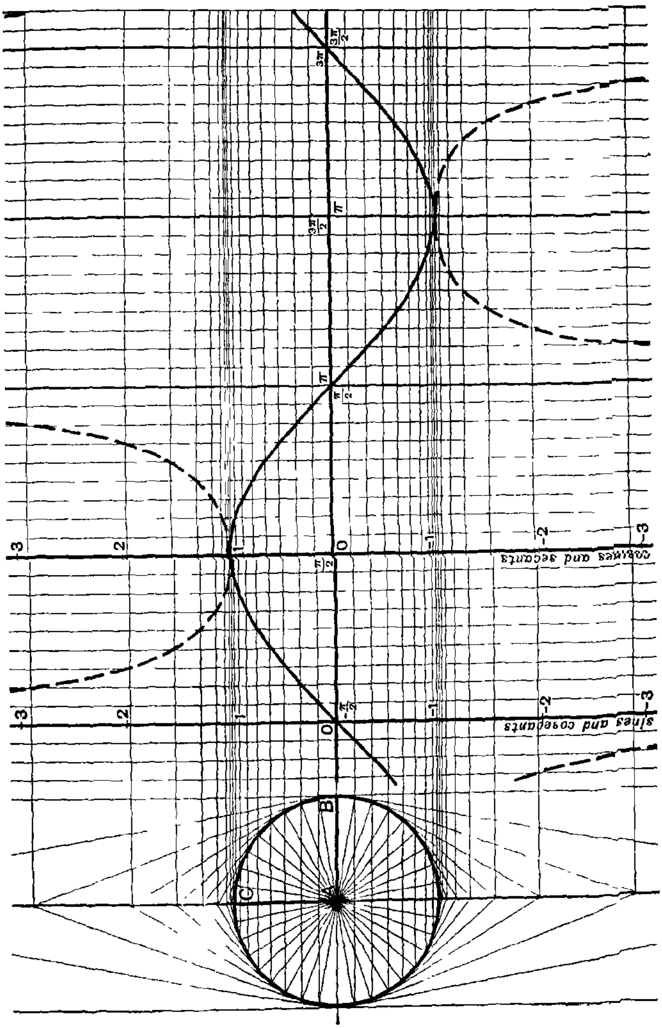


FIG. 88

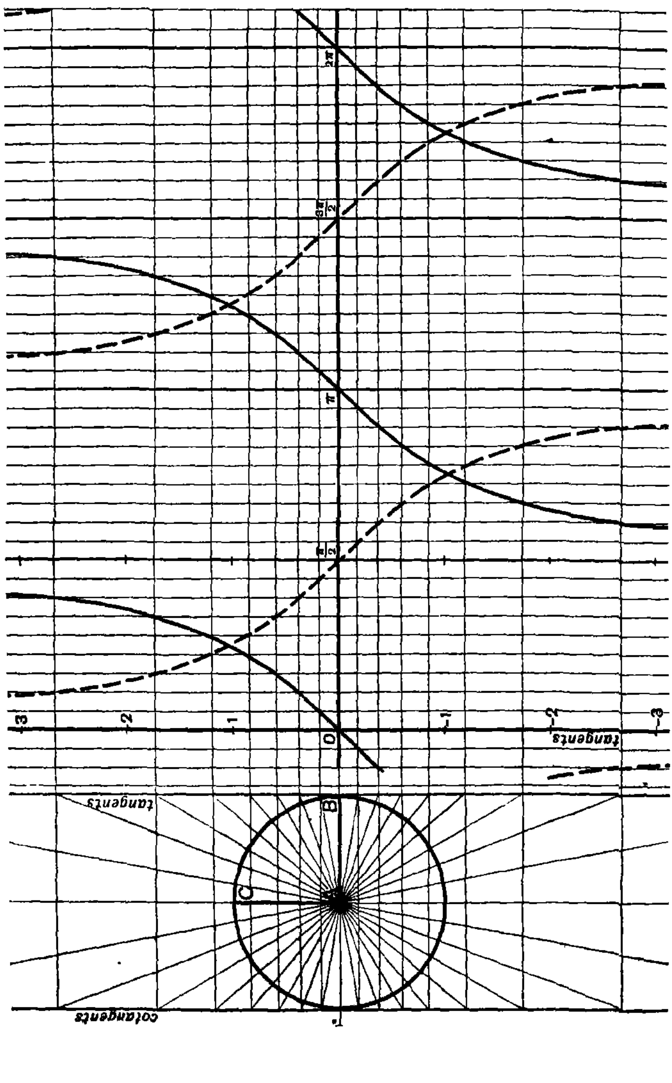


FIG. 89.

The variation and periodicity can best be shewn from a graph, *geometrically drawn*, i.e. not plotted from tables. Referring to the diagram on p. 45, the method used in drawing the graphs on pp. 173, 174 is obvious for the *sine* and *cosecant*: for the *cosine* and *secant*, in order to project horizontally the values of these ratios the diagram must be given a twist of $+\frac{\pi}{2}$; or, in other words, **for the complementary ratios AC must be chosen as the initial line instead of AB**: for the tangent and cotangent the diagram is slightly altered, these ratios being represented by lengths of lines on fixed tangents instead of variable ones. The student should draw the two kinds of diagrams for each of several angles lying in different quadrants and compare them, verifying that the sign and magnitude of the ratios agree with one another whichever diagram is used.

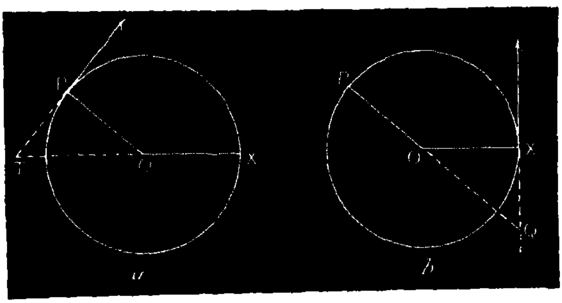


Fig. 90.

Thus, for the tangents of obtuse angles in Fig. 90 (a), the tangent **PT**, whose direction of measurement is shewn by the arrow-head, has to be produced **backward** to meet **OX** also produced backward: and in Fig. 90 (b) **OP** produced backward meets the tangent at **X** produced in the **negative direction.** Also the triangles **OPT** in (a) and

OXQ in (*b*) are congruent: hence the values of the tangents from the two diagrams are equal in magnitude and sign.

The graphs on pp. 173, 174, if accurately drawn, suggest nearly all the important relations between angles of any magnitude having equal or complementary ratios.

The method of construction at once shows a Periodicity of 2π for *sin*, *cos*, *sec*, *cosec*; and a Periodicity of π for the *tan* and *cot*. This can be further verified by taking a trace of any part of any of the graphs, and shifting the trace forward or backward through a distance corresponding to 2π parallel to the horizontal axis (i.e. the *length* of the page in the graphs given).

For the experiments which follow the student should draw (on tracing linen if possible) graphs of the ratios on as large a scale as is convenient. In order to obtain the true shapes of the graphs, $y = \sin x$, $y = \cos x$ etc., which is desirable, the radius of the circle must be so taken that

$$\frac{\text{radius of circle}}{\text{length representing } 90^\circ} = \frac{1}{\frac{\pi}{2}} = \frac{2 \times 113}{355}.$$

Thus in the graphs on pp. 173, 174, $\frac{1}{10}$ in. was taken to represent 10°, and hence the radius of the circle had to be

$$\frac{\cdot 9 \times 2 \times 113}{355} \text{ in.}$$
$$= \cdot 57 \text{ in. nearly.}$$

Each separate graph should first of all be examined for symmetry. The two kinds of symmetry usually considered are (1) symmetry about a line, (2) symmetry about a point.

Symmetry about a line exists when the figure can be folded about that line so that all points and lines of one part are brought into coincidence with corresponding points and lines of the other part (see p. 3).

Symmetry about a point (O) exists when to each point of the figure there corresponds another point so that the line joining them is bisected at O. In the case of symmetry round a point, one half the figure can be brought into coincidence with the other half, by rotation of the first half through an angle of two right angles: or part of the figure may be folded about any straight line through the pole of symmetry and again folded about a straight line through the pole at right angles to the first, and thus brought into coincidence with the other part. It should be observed that folding about the axis of x changes the signs of all the y's; folding about the axis of y changes the signs of all the x's; whilst folding about both axes (in case of symmetry with regard to a point) changes the signs of both x's and y's.

93. Consider the graph of $y=\sin x$, drawn true to scale and as large as possible, verify by folding (or by reference to the "circle of construction") that

(i) The graph is not symmetrical with regard to the axis of x, or any line parallel to it. Hence for each value of x there is only one value for the *sine*.

(ii) The curve is symmetrical with regard to the lines $x=(2n+1)\frac{\pi}{2}$, where n is any integer positive or negative.

Hence the values of the sines of angles equidistant from $(2n+1)\frac{\pi}{2}$ are equal; in particular $\sin A=\sin(\pi-A)$, these being equidistant from $\frac{\pi}{2}$.

(iii) The curve is symmetrical with regard to the pts. $x=n\pi$, $y=0$ where n is an integer positive or negative.

Hence the values of the sines of angles equidistant from $n\pi$ are equal in magnitude but opposite in sign; in particular

$$\sin(-A)=-\sin(+A),$$
$$\sin(\pi+A)=-\sin(\pi-A)=-\sin A \text{ by (ii).}$$

94. Consider the graphs for $\cos x$, $\tan x$ and deduce from the symmetry of these curves that

$$\begin{aligned}
&(1)\quad \cos(-A) = +\cos A,\\
&(2)\quad \cos(\pi - A) = -\cos A,\\
&(3)\quad \cos(\pi + A) = -\cos A,\\
&(4)\quad \tan(-A) = -\tan A,\\
&(5)\quad \tan(\pi - A) = -\tan A,\\
&(6)\quad \tan(\pi + A) = +\tan A.
\end{aligned}$$

95. Shew that the graphs of $\cos x$ and $\sin x$ are identical in form, if the graph of the cosine is shifted forward a distance representing $x=\frac{\pi}{2}$. Hence deduce that for an angle of any magnitude whatever

$$\sin\left(\frac{\pi}{2}+A\right)=\cos A,$$

hence

$$\begin{aligned}
\sin\left(\frac{\pi}{2}-A\right)&=\cos(-A)\\
&=\cos A \text{ by Expt. 94.}
\end{aligned}$$

96. Superimpose the graphs of the tangent and cotangent and, observing that the graph for the cotangent is the image of the graph for the tangent in any one of the lines

$$x=(2n+1)\frac{\pi}{4},$$

deduce that

$$\cot\left(A+\overline{2n+1}\,\frac{\pi}{4}\right)=\tan\left(\overline{2n+1}\,\frac{\pi}{4}-A\right)$$

and in particular

$$\cot A=\tan\left(\frac{\pi}{2}-A\right).$$

† **97. Find a formula for all the angles which have the same sine as a given angle θ.**

From Expt. 93 we have

$$\sin A=\sin(\pi-\theta).$$

Hence, since the sine is a periodic function of period 2π,

$$\begin{aligned}
\therefore\ \sin A&=\sin(2n\pi+\theta)\\
\sin(\pi-A)&=\sin(\overline{2m+1}\,\pi-\theta).
\end{aligned}$$

Now $(-1)^r$, where r is any integer, is equal to $+1$ or -1 according as r is even $(=2n)$ or odd $(=2m+1)$: hence the angles $2n\pi+\theta$, $\overline{2m+1}\,\pi-\theta$ are all included in the formula

$$r\pi+(-1)^r\theta.$$

$$\therefore\ \sin A=\sin(r\pi+(-1)^r\theta),$$

and the right-hand side of this equation includes *all* the values of the angle having the same sign as A, i.e.

$$\sin^{-1}(\sin\theta)=r\pi+(-1)^r\theta.$$

98. Verify the formula obtained for $\sin^{-1}(\sin A)$, by direct reference to a figure showing all these angles, and shew that there are no other values besides those given by the formula.

99. Shew, (i) by reference to the graph, (ii) by reference to a figure shewing all the angles, that the corresponding formulae for cosines and tangents are

$$\cos^{-1}(\cos\theta)=2n\pi\pm\theta,$$
$$\tan^{-1}(\tan\theta)=n\pi+\theta.$$

EXERCISES.

1. Shew that

(*a*) $\cos(180°+A)=\cos(A-180°)$.

(*b*) $\tan(90°+A)=\cot(180°-A)$.

(*c*) $\text{cosec}(180°+A)=\text{cosec}(-A)$.

2. Find an expression for all the angles satisfying the equation

$$\tan\theta=\cot\theta.$$

3. It can be shewn that

$$2\sin\frac{\theta}{2}=\pm\sqrt{1+\sin\theta}\pm\sqrt{1-\sin\theta}.$$

Examine this formula to determine which of the ambiguous signs must be taken as θ increases from -2π to $+2\pi$.

4. An angle is known to lie between 540° and 630°: its sine is given $=-\frac{1}{2}$; find the sine and cosine of half the angle.

5. The tangent of an angle between 270° and 360° is $-\frac{3}{4}$: find its sine and cosine.

6. Shew that the signs of the ratios *sin*, *cos*, and *tan* are never all the same for any pair of quadrants; also that if any two of these ratios are given with their proper signs the magnitude of the angle $(\pm 2n\pi)$ can be determined by a geometrical construction.

7. Find $\sin 37° 20'$ from the tables, and calculate its cosine: hence find (*a*) $\sin 127° 20'$; (*b*) $\cos 142° 40'$; (*c*) the sine, cosine and tangent of $217° 20'$, $307° 20'$, showing the triangle OPM in a diagram for each case.

8. From the tables it is found that

$$\sin 21° 20' = 0·36370 \text{ and } \cos 21° 20' = 0·93148;$$

find A (less than four right angles) when

$$\sin A = -0·36379 \text{ and } \cos A = -0·93148;$$

also find B (less than four right angles) when

$$\sin B = 0·93148, \ \cos B = -0·36379.$$

9. Find the eight positive values of θ (less than four right angles) for which

$$2\sin^2 2\theta = 1.$$

10. Examine the signs of the ratios *sin*, *cos*, and *tan*, for angles lying in the four quadrants, and tabulate the results.

Quad^t.	I	II	III	IV
sin				
cos				
tan				

11. Give verbal statements of the changes in sign and magnitude of (i) $\sin A$, (ii) $\cos A$, (iii) $\tan A$ as A increases from 0° to 360°.

12. *Draw a graph for

$$y = a \sin (vx + \beta)$$

where a, v, β are constants, and x is the circular measure of a variable angle.

* This is an important graph in problems on Sound, Cranks, and other branches of Higher Applied Mathematics.

TEST PAPERS (*continued*).

11.

[Board of Education, May 1903: Stage II., Trigonometry.]

1. (*a*) Define the logarithm of a number to a given base, and state what are the practical advantages of taking 10 as the base of a system of logarithms.

(*b*) Explain how it is that $\log(a^3)=3\log a$.

(*c*) Using the given table*, find log 36, and the numerical value of the fifth root of 1·125. Find also the numerical value of $(\tan 35^\circ 40')^3$.

2. (*a*) Find from a diagram the numerical value of the sine of 60°, and of the tangent of 60°. Find from your answer the tabular logarithm of the sine of 60°.

(*b*) Given $\sin 21^\circ 20'=0{\cdot}3638$ and $\cos 21^\circ 20'=0{\cdot}9315$, find A (less than four right angles) when

$$\sin A = -0{\cdot}3638 \text{ and } \cos A = -0{\cdot}9315.$$

Also, find B (less than four right angles) when

$$\sin B = 0{\cdot}9315 \text{ and } \cos B = -0{\cdot}3638.$$

3. (*a*) If A is an acute angle, shew by means of a diagram:

(i) $\cos(90^\circ - A)=\sin A$,

(ii) $\cos(90^\circ + A) = -\sin A$,

(iii) $\tan(180^\circ + A)=\tan A$.

(*b*) Shew that:

(i) $2(1+\sin A)(1+\cos A)=(1+\sin A+\cos A)^2$,

(ii) $\tan^2 A - \tan^2 B = \dfrac{\text{soc}^2 B - \cos^2 A}{\cos^2 A \cos^2 B}$.

4. Shew that in any plane triangle the sides are proportional to the sines of the opposite angles. N.B.—Two cases are to be considered.

ABC is a right-angled triangle, and the line bisecting the right angle C cuts AB in D; shew that

$$DB = AD \tan A.$$

* The requisite logarithms are supplied with these examination papers.

5. The hypotenuse of a right-angled triangle is 1,000 feet long, and the difference between the other two sides is 240 feet; calculate the other sides and angles of the triangle, and verify your result by drawing the triangle to scale.

6. P and Q are two stations 1,000 yards apart on a straight stretch of sea-shore, which bears East and West. At P a rock bears 42° West of South, at Q it bears 35° East of South. Shew that the distance of the rock from the shore is $1{,}000 \sin 48° \sin 55° \div \sin 77°$ yards, and calculate this distance to the nearest yard.

12.

[Board of Education, May 1904: Stage II., Trigonometry.]

1. Define the base of a system of logarithms.

State what number is taken as the base of the common logarithms.

Mention two properties of common logarithms which depend on the fact that that number is taken as their base.

Find by logarithms the numerical values of:

(*a*) $(1{\cdot}05)^{20}$;

(*b*) $(\frac{3}{7})^{\frac{1}{2}}$;

(*c*) $(\sin 18° 42')^3$.

2. (*a*) Find, by the aid of a diagram, the sine, cosine, and tangent of 30°.

(*b*) Write down the values of cos 150°, tan 210°, sin 330°.

(*c*) If $\tan A = \frac{4}{7}$, find the true logarithm of tan A, the tabular logarithm, and the number of degrees with the odd minutes and seconds in the angle A.

3. (*a*) The sine of an angle less than a right angle is 0·4; find the cosine, tangent, and cotangent of that angle.

(*b*) A is a positive angle less than four right angles, and it is given that

$$2 \sin A = \pm\sqrt{3};$$

find all the values of A.

(*c*) It is given that $\sin 32° = 0·530$ and $\cos 32° = 0·848$.

If $\cos B = 0·530$ and $\sin B = -0·848$, where B is a positive angle less than four right angles, explain why there can be only one value of B, and find it.

4. (*a*) If $\tan\theta + \text{cotan}\,\theta = 3$, find $\tan\theta$.

(*b*) Find the value of

$$\sqrt{(\text{cotan}^2\theta - \cos^2\theta)},$$

when $$(a+b)\sin\theta = a - b.$$

5. (*a*) In any triangle ABC shew that

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

(*b*) A, B, C are points on the circumference of a given circle; AD is drawn to meet the tangent at B at right angles in D, and AE is drawn to meet the tangent at C at right angles in E; shew that

$$\frac{AD}{AE} = \frac{AB^2}{AC^2}.$$

6. (*a*) In the triangle ABC, given $A = 40°$, $a = 7$ units, $c = 10$ units, find the two possible values of C and the corresponding values of B; and using the two values of B with the data given, find the two values of b.

(*b*) Construct the two different triangles ABC, in which $A = 40°$, $a = 7$ units, $c = 10$ units and write down the measure of the angle C in each case as read off from your protractor.

13.

[Board of Education, May 1905: Stage II., Trigonometry.]

1. (*a*) Define a logarithm.

Reasoning from your definition, find the logarithm of 81 to the base 3, and the logarithm of $\frac{1}{4}$ to base 2.

(*b*) Find by means of the tables the common logarithms of

(i) 0·00144, (ii) $(\tan 34° 47')^{\frac{1}{5}}$.

(*c*) Find the numerical value of

$$(6)^{\frac{1}{3}} \times (12)^{\frac{1}{3}} \div (125)^{\frac{1}{4}}.$$

2. (*a*) Find, with the aid of an appropriate diagram, the sine, cosine, and tangent of an angle of 30°, and also of an angle of 60°. Write down with proper signs the numerical values of the following expressions:

$$\sin 120°, \quad \cos 150°, \quad \tan 150°.$$

(*b*) The angular elevation of the top of a spire is 60°, and the angular elevation of the top of the tower on which the spire stands is 30°; the height of the tower is 50 feet; calculate the height of the spire, and verify your result by using a scale and protractor.

3. (*a*) The ratio of two geometrical magnitudes is commonly denoted by the letter π; state what those magnitudes are.

Without going into much detail, indicate why you believe that π is nearly equal to 3·14159.

(*b*) Find the number of degrees, with the odd minutes and seconds, in the angle subtended at the centre of a circle by an arc twice as long as the radius.

4. Establish the formula

$$\left(\sin\frac{A}{2}\right)^2 = \frac{(s-b)(s-c)}{bc},$$

and by means of it find the greatest angle of the triangle whose sides are 13 feet, 30 feet, and 37 feet long respectively.

5. Find an expression for the area of a triangle (i) when two sides and the included angle are given, (ii) when one side and the angles are given.

A and C are two points on a given straight line, and two parallel lines AB, CD are drawn on the same side of AC, X is a point in AC, Y a point in AB, Z a point in CD, shew that twice the area of the triangle XYZ equals

$$(CX \cdot AY + AX \cdot CZ) \sin BAC$$

6. ABC is a triangle having a right angle at A, let D be the middle point of BC, and E the point in which the line bisecting the right angle cuts BC, shew that

$$2DE = BC \tan(45^\circ - B).$$

If we suppose that a circle is described about the triangle ABC, explain how the point E moves along BC, when A moves from B to C along the circumference of the circle.

14.

[Board of Education, May 1906 Stage II, Trigonometry]

1. Explain why the logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers

By means of logarithms given below, find the fifth root, and the fifth power of 0 69889 correct to five decimal places

If $4 \log_{10} x + 7 = 0$, find x

If $y \log_{10} 0\,0424 = \log_{10} 0\,2165$, find y to four places of decimals

2. Draw an appropriate diagram, and from it find the numerical values of the sine, cosine, and tangent of an angle of 45°

Find also the true logarithm and the tabular logarithm of sin 45° and of tan 45°.

If A be any angle between 0° and 90°, find tan A in terms of sin A, and also in terms of cos A

3. Shew, in a carefully drawn diagram, an angle 234°, and explain, with reference to your diagram, why both the sine and the cosine of that angle are negative.

Assuming that $\sin 36° 56'$ is 0·6, and that $\cos 36° 56'$ is 0·8, find the angle whose sine is $-0·6$ and whose cosine is $+0·8$.

Find, from the annexed table, the numerical value of $\sin 326° 42'$.

4. Establish the following identities:—

(*a*) $\sin^4 A + \cos^2 A = \cos^4 A + \sin^2 A.$

(*b*) $\sin^2 A \tan^2 A = \tan^2 A - \sin^2 A.$

(*c*) $$\sin A \cos A = \frac{\tan A}{1 + \tan^2 A}.$$

(*d*) $$\frac{\cos^2 A - \sin^2 B}{\sin^2 A \sin^2 B} = \frac{1}{\tan^2 A \tan^2 B} - 1.$$

5. Two points A and B are 2,000 yards apart on a straight road, and P is a flagstaff off the road; it is found that the angles PAB and PBA are 33° 18′ and 105° 20′ respectively.

Calculate the distance BP, and the number of square yards in the triangle ABP.

6. Shew that the area of a quadrilateral is equal to the area of a triangle having two sides equal to the diagonals of the quadrilateral, and the included angle equal to either of the angles between the diagonals.

Find the area of the quadrilateral in which the diagonals are 216·5 ft. and 447·5 ft. long respectively, and are inclined to each other at an angle of 116° 30′.

ANSWERS.

PAGE

12. 2. 120°. 3. (i) $\frac{1}{2}^\circ$, (ii) $6''$. 4. (i) $17\frac{1}{2}^\circ$, (ii) 70°, (iii) $129\frac{1}{2}^\circ$. 7. 16° 40′ 0″. 8. 21. 9. 43° 19′ 51″, 86° 39′ 42″.

13. 1. $11\frac{1}{4}^\circ$. 2. $56\frac{1}{4}^\circ$, $56\frac{3}{4}^\circ$. 3. B = 119°, C = 117°, D = 84°, E = 159°, F = 61°. [Approx.]

14. 5. C, $2\frac{1}{4}$ miles, S. by 19° W.; D, 5 miles, W. by 31° S.; E, $4\frac{1}{2}$ miles, W. by 15° N.; F, $5\frac{1}{8}$ miles, N. by 42° W. [Approx.] 6. $13\frac{3}{4}$ miles. [Approx.] 7. 21 knots, S. by 37° E. 8. 108 yds.

21. 1. 3 ft. $6\frac{1}{5}$ in 2. $55\frac{1}{5}^\circ$. 3. 117 ft.

24. 4. $28\frac{1}{2}$ ft. 5. 1219 ft., 361 ft 6. 132 ft., 119 ft. **Expt. 19.** 212 ft.

25. 7. 639 yds. 8. $5\frac{1}{2}^\circ$, $16\frac{1}{2}^\circ$. 9. 0·12 in., 2·65 in. 10. 67 ft. 11. 14 ft. 12. 50 ft. 9 in. 13. 127 ft. 14. 147 ft.

42. 1. Let ABCD be ▱$^{\text{m}}$, AB = 15, BC = 24. Draw AE and DF ⊥ BC. Find, in succession, AE, BE, CE, ∠ACB, ∠DBF: hence AC = 19·2, BD = 33. 2. 65 ft. 3. 28·2 sq. feet. 4. 125 ft. 5. 280 ft. 6. $891\frac{1}{2}$ ft.

43. 7. 154 ft. 8. 11° 5′. 9. 2329 yds. 10. $87\frac{1}{2}$ seconds. 11. 1383 ft. 12. 116 ft.

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47.

If	sin A	cos A	tan A	cot A	sec A	cosec A
sin A $= s$	s	$\sqrt{1-s^2}$	$\dfrac{s}{\sqrt{1-s^2}}$	$\dfrac{\sqrt{1-s^2}}{s}$	$\dfrac{1}{\sqrt{1-s^2}}$	$\dfrac{1}{s}$
cos A $= c$	$\sqrt{1+c^2}$	c	$\dfrac{\sqrt{1-c^2}}{c}$	$\dfrac{c}{\sqrt{1-c^2}}$	$\dfrac{1}{c}$	$\dfrac{1}{\sqrt{1-c^2}}$
tan A $= t$	$\dfrac{t}{\sqrt{1+t^2}}$	$\dfrac{1}{\sqrt{1+t^2}}$	t	$\dfrac{1}{t}$	$\sqrt{1+t^2}$	$\dfrac{\sqrt{1+t^2}}{t}$
cot A $= x$	$\dfrac{1}{\sqrt{1+x^2}}$	$\dfrac{x}{\sqrt{1+x^2}}$	$\dfrac{1}{x}$	x	$\dfrac{\sqrt{1+x^2}}{x}$	$\sqrt{1+x^2}$
sec A $= y$	$\dfrac{\sqrt{y^2-1}}{y}$	$\dfrac{1}{y}$	$\sqrt{y^2-1}$	$\dfrac{y}{\sqrt{y^2-1}}$	y	$\dfrac{y}{\sqrt{y^2-1}}$
cosec A $= z$	$\dfrac{1}{z}$	$\dfrac{\sqrt{z^2-1}}{z}$	$\dfrac{1}{\sqrt{z^2-1}}$	$\sqrt{z^2-1}$	$\dfrac{z}{\sqrt{z^2-1}}$	z

48. 1. $\frac{3}{4}$, $\frac{5}{3}$. 2. $\dfrac{2\sqrt{2}}{3}$, $\dfrac{1}{2\sqrt{2}}$. 3. $\frac{4}{5}$, $\frac{5}{3}$. 4. $\dfrac{1}{\sqrt{15}}$, $\dfrac{\sqrt{15}}{4}$.

5. $\dfrac{\sqrt{3}}{2}$, $\dfrac{1}{2}$. 6. $\dfrac{\sqrt{5}}{3}$, $\dfrac{3}{2}$. 7. $\dfrac{b}{\sqrt{c^2-b^2}}$. 8. $\dfrac{a}{\sqrt{a^2+b^2}}$, $\dfrac{b}{\sqrt{a^2+b^2}}$. 9. $\dfrac{\sqrt{a^2-1}}{a}$, $\dfrac{1}{\sqrt{a^2-1}}$. 11. $h^2(1+k^2)=1$.

59. 1. (a) $\begin{matrix}\sin=0{\cdot}19156\\ \tan=0{\cdot}19517\end{matrix}\begin{bmatrix}0{\cdot}1915562\\0{\cdot}1951703\end{bmatrix}$*, $\begin{matrix}\sin=0{\cdot}63415\\ \tan=0{\cdot}82016\end{matrix}\begin{bmatrix}0{\cdot}6341533\\0{\cdot}8201597\end{bmatrix}$.

(b) $\begin{matrix}\cos=0{\cdot}56712\\ \text{cosec}=1{\cdot}21413\end{matrix}\begin{bmatrix}0{\cdot}5671212\\1{\cdot}2141310\end{bmatrix}$, $\begin{matrix}\cos=0{\cdot}13702\\ \text{cosec}=1{\cdot}00952\end{matrix}\begin{bmatrix}0{\cdot}1370220\\1{\cdot}0095217\end{bmatrix}$.

(c) 1·00000 [1·0000003], 1·05994 [1·0599366].

(d) 0·54660 [0·5466022], 0·19044 [0·1904387].

2. (a) 30°, 60°, 15°, 45°, 18°.

(b) 14° 28′ 39″ [14° 28′ 39″], 19° 28′ 15″ [19° 28′ 16″], 77° 27′ 47″ [77° 27′ 49″], 35° 15′ 50″ [35° 15′ 52″].

(c) 71° 33′ 53″ [71° 33′ 54″], 30° [30°], 60° [60°], 19° 28′ 17″ [19° 28′ 16″].

[* The numbers in brackets have been obtained from seven figure tables, and are inserted for the sake of comparison with the results obtained from the five figure tables on pages 36—41.]

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63. 7. (1) 210; (2) 1872; (3) 156, $\cos C = -\frac{2}{15}$ $\therefore$ C is obtuse.

70. 2. $\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$, $\sin 75^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$,

$\cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$, $\cos 75^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$,

$\tan 15^\circ = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$; $\tan 75^\circ = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 2+\sqrt{3}$.

71. 4. 45°, 60°, 75°. 5. 120°. 6. A = 54° or 126°, B = 108° or 36°, $b = \sqrt{10+2\sqrt{5}}$. 7. $a = 2(\sqrt{3}-1)$, $c = 2\sqrt{2}$. 8. $50\sqrt{3}$. 9. C = 75° or 105°, A = 90° or 60°, $a = 2\sqrt{2}$ or $\sqrt{6}$. 10. $5\sqrt{3} : 8 : 4+3\sqrt{3}$; the sine of the third angle is found by the formula in Ex. 5, p. 63.

11. 47° 41′ 15″, 38° 56′ 32″ : hence the third angle is 93° 22′ 23″, although the calculated value is 93° 22′ 22″; see p. 97. 12. 169·7. 13. 7·49 in., 8·58 in.

72. 14. 74·6 ft. 15. 42° 19′ 4″. 16. AM = 23,400 yds., BM = 23,280 yds. 17. 10·6. 18. 226 ft. 20. 6000 ft.

77. 1. (*a*) 1·66; (*b*) 0·3; (*c*) 2·35; (*d*) 4·71. 2. 1·732; 2·236; 2·45; 2·644. 3. 1·26; 1·587; 2·08.

79. 2. 0·60206; 0·77815; 0·90309; 1·07918; 1·20412; 1·25527. 4. 1·43136; 1·50515; 1·80618; 1·90849. 6. 0·15052; 0·15904; 0·31808; 0·53959. 9. 0·69897; 1·17609; 0·17609; 0·87506; 0·09691; 0·55630; 0·68124.

80. 10. 0, 1, 2, 3. 11. 2·09684 by interpolation: 2·09691 by factors: hence the ratio $\dfrac{126-120}{120}$ is too large to allow the rule of proportional differences to give a trustworthy figure in the fifth decimal place. It will be found on reference to the tables that $2{\cdot}09684 = \log_{10} 124{\cdot}98$ instead of $\log_{10} 125$. 12. (*a*) 2·00328. (*b*) 2·00329: seven-figure tables give 2·0032882. 13. By interpolation we get 4·39946 instead of 4·39947 from the factors: seven-figure tables give 4·3994660.

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81. 14. 1·53466, $\bar{2}$·53466.

82. 15. 3·672, 0; $2·3075 \times 10^2$, 2; $2·3 \times 10^{-2}$, $\bar{2}$; 3×10^{-1}, $\bar{1}$; 1×10^{-5}, $\bar{5}$. 16. The powers of ten appearing as factors when the numbers are reduced to standard form are respectively 10^9, 10^8, 10^{-1}, 10^0.

86. 5. 0·33491. 6. 0·1118. 7. 428270. 8. 0·076539. 9. 0·65072. 10. 2·6534. 11. 0·80911. 12. 0·00011708. 13. 59·6. 14. 0·99178. 15. $\bar{1}$·56941. 16. $\bar{1}$·79022. 17. 0·01223. 18. 1·1063. 19. 1·6107.

97. 3. (i) 8·5889226; 9·0507984; 9·9923305; 9·9996048.
(ii) 10·2449297; 10·7327623; 10·0296687; 11·0918799.
(iii) 10·0870962; 10·0037598; 10·2793620; 10·9221373.

98. 4. 2·0024685; 3·0001303; $\bar{1}$·0093488; 2·0159965; $\bar{2}$·0345321; 0·0000087; 3·0409041.

99. 5. (1) 2·9541074; 2·9506569; $\bar{1}$·8594127; 0·9924117.
(2) 971·941; 8921·748; ·899477.

108. 4. $r=4$, $r_1=24$, $r_2=12$, $r_3=8$, $R=10$.
5. (i) 38° 52′ 48″; 67° 22′ 51″; 73° 44′ 25″; 204.
[38° 52′ 48·2″] [67° 22′ 48·5″] [73° 44′ 23·3″]
(ii) 112° 37′ 9″; 36° 52′ 13″; 30° 30′ 39″; 66.
[112° 37′ 11·5″] [36° 52′ 11·6″] [30° 30′ 36·9″]
(iii) 143° 7′ 47″; 20° 36′ 36″; 16° 15′ 38″; 462.
[143° 7′ 48·4″] [20° 36′ 34·9″] [16° 15′ 36·7″]
(iv) 75° 45′ 3″; 67° 22′ 51″; 36° 52′ 13″; 126.
[75° 44′ 59·9″] [67° 22′ 48·5″] [36° 52′ 11·6″]

N.B. The above results should be carefully compared in connection with the notes on pp. 110, 111.

5. 0·017 sq. in. 6. 26° 33′ 54″; 116° 33′ 54″; 36° 52′ 12″.

117. 6. 23 ac. 2 rd. 16 pl. 7. 74° 58′ 38″. 8. B = 98° 3′ 53″, C = 30° 56′ 7″. 9. B = 45° 23′ 28″, C = 99° 0′ 12″, c = 3004·2.
10. (i) 9° 57′ 37″, 151° 23′ 23″, 32·62.
(ii) 132° 46′ 23″, 28° 36′ 37″, 137·96.

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118. **11.** (i) Included angle $= 41° 47' 57''$, remaining side $=$ 201 4 ft
(ii) Included angle $= 138° 12' 3''$, remaining side $=$ 468 4 ft
12. 8578 yds **13** 47 7 ft, 63 9 ft. **14.** 317 6 ft
15 134 8 ft **16** (1) 5355 ft, (2) $6° 10'$, (3) $48° 4' 51'$, (4) 16929 yds

120. **1** 29 462 in, 748 3 mm

126. **3** Divide the scale into degrees and thirds take arc on vernier equal to 13° and divide into 40 equal parts **4.** Radius of arc about 12 in, divided into intervals of 3′, 179 of these divisions are divided, on the vernier, into 180 equal parts

TEST PAPERS

N B *Seven figure tables have been used in the calculation of the answers to these test papers*

138. **1.** **1** (a) $70\frac{5}{16}$, (b) 81° **2** $22\frac{1}{2}°$ **3** 12 52 p m
4 10 in very nearly

2. **1** 11 miles nearly **2** $\frac{56}{65}$, $\frac{33}{65}$, $\frac{56}{33}$

139. **3** (i) $\frac{24}{25}$, (ii) 90°, (iii) 1, (iv) $\sqrt{1-c^2}$, $\sqrt{1-s^2}$, $\frac{s}{\sqrt{1-s^2}}$, $\frac{t}{\sqrt{1+t^2}}$, $\frac{1}{\sqrt{1+t^2}}$, $\frac{\sqrt{1-c^2}}{c}$

3. **2** $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\theta = 53° 7' 48''$

3 $\sin\theta = \frac{c(b-a)}{a^2+b^2}$, $\cos\theta = \frac{c(b+a)}{a^2+b^2}$ **4.** $50° 36'$.

140. **4.** **1** 225 ft Draw a vertical line **TF**, and a horizontal line **TH** With the protractor set off **TX**, **TY**, **TZ** making angles 34°, 36°, 40° with the horizontal **TH**; join **XY**, produce **XY** to **W**, making **YW** $=$ **XY**, draw **WV** $\parallel$ **YT** to meet **TZ** in **V**, join **XUV** and draw **ABC** $\parallel$ **XUV**, so that **AB** $=$ **BC** $= 4''$ Through **A** draw **AF** $\parallel$ **HT** Then **TF** gives the height required on a scale of $1'' = 25$ ft

4 4 12″, $51° 41' 2''$

5. **1** $7\frac{1}{2}$ inches **3** (i) $\cos\theta = \frac{1}{2}$, $\theta = 60°$, (ii) $\tan\theta = \frac{1}{2}$ or $\frac{3}{4}$, $\theta = 56° 18' 35''$ or $36° 52' 12''$

141. **5** Velocity

6. **1.** 10 58 in, 26 61 in, 56 54 sq in **2.** (i) $\frac{12}{13}$, (ii) 2 291 **3.** 297 35 sq in, 309 49 sq in; 312 01 sq in **4** -1 **5** 14 36 ft

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142. **7.** 1. $\frac{1}{10\cdot 5}$, $\frac{1}{42\cdot 2}$, $\frac{1}{151}$ up; $\frac{1}{66}$, $\frac{1}{110}$ down; $\frac{1}{251}$, $\frac{1}{143}$, $\frac{1}{48}$ · $\frac{1}{48}$, $\frac{1}{132}$, $\frac{1}{293}$, $\frac{1}{121}$, $\frac{1}{406}$ up; $\frac{1}{27\cdot 3}$, $\frac{1}{39\cdot 6}$, $\frac{1}{17\cdot 4}$, $\frac{1}{278}$ down; $\frac{1}{1760}$, $\frac{1}{660}$ up; $\frac{1}{13\cdot 4}$ down.

N.B. *The technical term "gradient" as used by railway men indicates the* sine *of the angle of slope, and not the tangent.*

3. 11·716 miles. 4. $122\frac{1}{2}°$ about. 5. 30° 23′ 49″.

8. 2. 0·8571736, 3·499669, 21·30085.

143. 4. -1, -1, $\frac{1}{6}$, $\frac{8}{3}$, 3·32193.

9. 1. 1391·96. 2. $\log_{10} e = 0\cdot 4342942$. 3. 179266 sq. ft. 4. ·005 % nearly. 5. 24° 25′, 45° 35′.

144. **10.** 2. $\log 5 = 2a - c$, $\log 7 = c - a$, $\log 13 = b + 2c - 4a$. 4. 20° 37′, 89° 51′, 66·518 ft.; error in third side is less than $\frac{1}{100}$ in. 5. (*a*) 59 scale divisions divided into 60 equal parts on vernier; (*b*) zero of vernier between 20·2 and 20·3 of scale and division 6 of vernier opposite 20·8 on scale; where 9 scale divisions equal 10 vernier divisions.

THEORETICAL.

157. 1. 143° 14′ 22″. 2. 0·63274. 3. 3 ft. 4. 67 in. 5. 469 yds. 1 ft. 6. 24 minutes.

160. 1. Slightly more than 224. 2. $\frac{3\pi}{5}$, $\frac{2\pi}{3}$, $\frac{3\pi}{4}$, $\frac{4\pi}{5}$. 3. 57 in. 4. 62 in. 5. 3 : 8. 6. 6. 7. Area $= \frac{1}{4\pi} \times (\text{circumference})^2$; 45·7 in. 8. 4 ft. nearly. 9. 5·77 in.; 8·16 in. 10. £24. 1*s*. 3*d*.

171. **Expt. 92.** (1) $-0\cdot 86603$; (2) $+1$; (3) $-0\cdot 70711$; (4) $+0\cdot 86603$; (5) -1; (6) infinity.

179. 2. $(2n+1)\frac{\pi}{4}$. 4. $-\cdot 5$, $+0\cdot 86603$. 5. $-0\cdot 75$, $+0\cdot 8$. 7. (*a*) 0·79512; (*b*) 0·60645; (*c*) $-0\cdot 60645$, $-0\cdot 79512$, $+0\cdot 76272$; $-0\cdot 79512$, $+0\cdot 60645$, $-1\cdot 64894$. 8. 201° 20′; 111° 20′. 9. $\frac{\pi}{8}$, $\frac{3\pi}{8}$, $\frac{5\pi}{8}$, $\frac{7\pi}{8}$, $\frac{9\pi}{8}$, $\frac{11\pi}{8}$, $\frac{13\pi}{8}$, $\frac{15\pi}{8}$.

TEST PAPERS (*continued*).

(N.B. *Answers calculated with seven-figure tables.*)

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181. **11.** 1. (*c*) 1·5563025, 1·023835, 0·369669. 2. (*a*) 0·8660254, 1·7320508, 9·9375306; (*b*) See Ans. Ex. 8, p. 180.

182. 5. 576·85, 816·85, 35° 13′ 45·4″, 54° 46′ 14·6″.
6. 625 yds.

12. 1. (*a*) 26·533; (*b*) 0·8091064; (*c*) 0·0329415.
2. (*a*) $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{3}}$; (*b*) $-\frac{\sqrt{3}}{2}$, $+\frac{1}{\sqrt{3}}$, $-\frac{1}{2}$;
(*c*) $\bar{1}$·7569620, 9·7569620, 29° 44′ 42″.

183. 3. (*a*) 0·9165150, 0·4364361, 2·2912858: (*b*) 60°, 120°, 240°, 300°: (*c*) B = 202°.
4. (*a*)* $\tan\theta = \frac{3 \pm \sqrt{13}}{2} = 3{\cdot}3027756$ *or* 0·3027756:
(*b*) $\frac{4ab}{a^2 - b^2}$. 6. (*a*) C = 66° 40′ 28″ *or* 113° 19′ 32″, B = 73° 19′ 32″ *or* 26° 40′ 28″, $b = 10{\cdot}535$ *or* 4·889.

184. **13.** 1. (*a*) 4, −2; (*b*) $\bar{3}$·1583625, 0·893814; (*c*) 0·89329.
2. (*a*) $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{3}}$; $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\sqrt{3}$; $+\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $-\frac{1}{\sqrt{3}}$:
(*b*) 100 ft. 3. (*b*) 114° 35′ 30″. 4. 112° 37′ 14″.

185. 6. A graph of $DE = \frac{1}{2}\tan\left(\frac{\theta}{2} - 45^\circ\right)$ where θ is the angle BDA will shew the displacement; two derived curves for the velocity and acceleration should be drawn by plotting the "rate of increase" in each curve against θ to obtain the respective derived curves.

14. 1. 0·93085; 0·16674; 0·0177828; 0·4841.

186. 3. 323° 4′, −0·5490228. 5. 1153¼ yds., 1112201 sq. yds. 6. 433523 sq. ft

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